Computer Science Questions
(From the Oxford MAT Tests)

For answers, see the MAT website

Specimen A, Question 7:

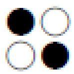
The game of *Oxflip* is for one player and involves circular counters, which are white on one side and black on the other, placed in a grid. During a game, the counters are flipped over (changing between black and white side uppermost) following certain rules.

Given a particular size of grid and a set starting pattern of whites and blacks, the aim of the game is to reach a certain target pattern. Each "move" of the game is to flip over either a whole row or a whole column of counters (so one whole row or column has all its blacks swapped to whites and vice versa). For example, in a game played in a three-by-three square grid, if you are given the starting and target patterns

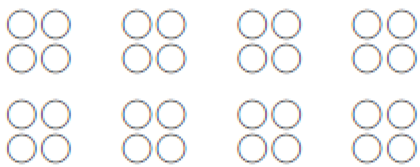
a sequences of three moves to achieve the target is:

There are many other sequences of moves which also have the same result.

(i) Consider the two-by-two version of the game with starting pattern



Draw, in the blank patterns below, the eight different target patterns (including the starting pattern) that it is possible to obtain.



What are the possible numbers of white counters that may be present in these target patterns?

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Specimen B, Question 7:

Suppose you have an unlimited supply of black and white pebbles. There are four ways in which you can put two of them in a row: BB, BW, WB and WW.

(i) Write down the eight different ways in which you can put three pebbles in a row.

(ii) In how many different ways can you put N pebbles in a row?

Suppose now that you are not allowed to put black pebbles next to each other. There are now only three ways of putting two pebbles in a row, because BB is forbidden.

(iii) Write down the five different ways that are still allowed for three pebbles.

Now let r_N be the number of possible arrangements for N pebbles in a row, still under the restriction that black pebbles may not be next to each other, so $r_2 = 3$ and $r_3 = 5$.

(iv) Show that for $N \geq 4$ we have $r_N = r_{N-1} + r_{N-2}$. Hint: consider separately the case where the last pebble is white, and the case where it is black.

Finally, suppose that we impose the further restriction that the first pebble and the last pebble cannot both be black. Let w_N be the number of such arrangements for N pebbles; for example, $w_3 = 4$, since the configuration BWB is now forbidden.

(v) For $N \geq 5$, write down a formula for w_N in terms of the numbers r_i, and explain why it is correct.

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2007, Question 7:

Suppose we have a collection of tiles, each containing two strings of letters, one above the other. A **match** is a list of tiles from the given collection (tiles may be used repeatedly) such that the string of letters along the top is the same as the string of letters along the bottom. For example, given the collection

$$\left\{ \left[\frac{\mathsf{AA}}{\mathsf{A}}\right], \left[\frac{\mathsf{B}}{\mathsf{ABA}}\right], \left[\frac{\mathsf{CCA}}{\mathsf{CA}}\right] \right\},$$

the list

$$\left[\frac{\mathsf{AA}}{\mathsf{A}}\right] \left[\frac{\mathsf{B}}{\mathsf{ABA}}\right] \left[\frac{\mathsf{AA}}{\mathsf{A}}\right]$$

is a match since the string AABAA occurs both on the top and bottom.

Consider the following set of tiles:

$$\left\{ \left[\frac{\mathsf{X}}{\mathsf{U}}\right], \left[\frac{\mathsf{UU}}{\mathsf{U}}\right], \left[\frac{\mathsf{Z}}{\mathsf{X}}\right], \left[\frac{\mathsf{E}}{\mathsf{ZE}}\right], \left[\frac{\mathsf{Y}}{\mathsf{U}}\right], \left[\frac{\mathsf{Z}}{\mathsf{Y}}\right] \right\}.$$

(a) What tile must a match begin with?

(b) Write down all the matches which use four tiles (counting any repetitions).

(c) Suppose we replace the tile $\left[\frac{\mathsf{E}}{\mathsf{ZE}}\right]$ with $\left[\frac{\mathsf{E}}{\mathsf{ZZZE}}\right]$.

What is the least number of tiles that can be used in a match?

How many different matches are there using this smallest numbers of tiles?

[Hint: you may find it easiest to construct your matches backwards from right to left.]

Consider a new set of tiles $\left\{ \left[\frac{\mathsf{XXXXXXX}}{\mathsf{X}}\right], \left[\frac{\mathsf{X}}{\mathsf{XXXXXXXXXX}}\right] \right\}$. (The first tile has seven Xs on top, and the second tile has ten Xs on the bottom.)

(d) For which numbers n do there exist matches using n tiles? Briefly justify your answer.

Computer Science Questions
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For answers, see (the MAT website)

2008, Question 7:

Ox-words are sequences of letters a and b that are constructed according to the following rules:

I. The sequence consisting of no letters is an Ox-word.

II. If the sequence W is an Ox-word, then the sequence that begins with a, followed by W and ending in b, written aWb, is an Ox-word.

III. If the sequences U and V are Ox-words, then the sequence U followed by V, written UV, is an Ox-word.

All Ox-words are constructed using these rules. The *length* of an Ox-word is the number of letters that occur in it. For example $aabb$ and $abab$ are Ox-words of length 4.

(i) Show that every Ox-word has an even length.

(ii) List all Ox-words of length 6.

(iii) Let W be an Ox-word. Is the number of occurrences of a in W necessarily equal to the number of occurrences of b in W? Justify your answer.

You may now assume that every Ox-word (of positive length) can be written *uniquely* in the form $aWbW'$ where W and W' are Ox-words.

(iv) For $n \geqslant 0$, let C_n be the number of Ox-words of length $2n$. Find an expression for C_{n+1} in terms of $C_0, C_1, \cdots, C_n$. Explain your reasoning.

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2009, Question 7:

Consider sequences of the letters M, X and W. *Valid* sequences are made up according to the rule that an M and a W can never be adjacent in the sequence. So M, XMXW, and XMMXW are examples of valid sequences, whereas the sequences MW and XWMX are not valid.

(i) Clearly, there are 3 valid sequences of length 1. List all valid sequences of length 2.

(ii) Let $g(n)$ denote the number of valid sequences of length n. Further, let $m(n)$, $x(n)$, $w(n)$ denote the number of valid sequences of length n that start with an M, an X, a W respectively.

Explain why

$$\begin{aligned} m(n) &= w(n), \\ m(n) &= m(n-1) + x(n-1) \quad \text{for } n > 1, \\ x(n) &= 2m(n-1) + x(n-1) \quad \text{for } n > 1, \end{aligned}$$

and write down a formula for $g(n)$ in terms of $m(n)$ and $x(n)$.

Hence compute $g(3)$, and verify that $g(4) = 41$.

(iii) Given a sequence using these letters then we say that it is *reflexive* if the following operation on the sequence does not change it: reverse the letters in the sequence, and then replace each occurrence of M by W and vice versa. So MXW, WXXM and XWXMX are reflexive strings, but MXM and XMXX are not. Let $r(n)$ be the number of valid, reflexive sequences of length n.

If a sequence is reflexive and has odd length, what must the middle letter be? Explain your answer.

Hence, show that

$$r(n) = \begin{cases} x\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ x\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$

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2010, Question 7:

In a game of *Cat and Mouse*, a cat starts at position 0, a mouse starts at position m and the mouse's hole is at position h. Here m and h are integers with $0 < m < h$. By way of example, a starting position is shown below where $m = 7$ and $h = 12$.

With each turn of the game, one of the mouse or cat (but not both) advances one position towards the hole *on the condition that the cat is always strictly behind the mouse and never catches it.* The game ends when the mouse reaches the safety of its hole at position h.

This question is about calculating the number, $g(h, m)$, of different sequences of moves that make a game of Cat and Mouse.

Let C denote a move of the cat and M denote a move of the mouse. Then, for example, $g(3, 1) = 2$ as MM and MCM are the only possible games. Also $CMCCM$ is *not* a valid game when $h = 4$ and $m = 2$ as the mouse would be caught on the fourth turn.

(i) Write down the five valid games when $h = 4$ and $m = 2$.

(ii) Explain why $g(h, h-1) = h - 1$ for $h \geqslant 2$.

(iii) Explain why $g(h, 2) = g(h, 1)$ for $h \geqslant 3$.

(iv) By considering the possible first moves of a game, explain why

$$g(h, m) = g(h, m+1) + g(h-1, m-1) \qquad \text{when } 1 < m < h-1.$$

(v) Below is a table with certain values of $g(h, m)$ filled in. Complete the remainder of the table and verify that $g(6, 1) = 42$.

m					
5					5
4				4	
3			3		
2		2			
1	1				
	2	3	4	5	6	h

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2011, Question 7:

Alice and Bob have a large bag of coins which they use to play a game called HT-2. In this game, Alice and Bob take turns placing one coin at a time on the table, each to the right of the previous one; thus they build a row of coins that grows to the right. Alice always places the first coin. Each coin is placed head-up (H) or tail-up (T), and cannot be flipped or moved once it has been placed.

A player loses the game if he or she places a coin that results in two adjacent coins having the same pattern of heads and tails as another adjacent pair somewhere in the row (reading a pattern from left to right). For example, Bob has lost this game by producing a second instance of HT (where a and b denote coins placed by Alice and Bob respectively):

$$\begin{array}{cccccc} a & b & a & b & a & b \\ H & H & T & T & H & T \end{array}$$

and Alice has lost this game by producing a second instance of TT (overlapping pairs can count as repeats):

$$\begin{array}{ccccc} a & b & a & b & a \\ T & H & T & T & T \end{array}$$

(i) What is the smallest number of coins that might be placed in a game of HT-2 (including the final coin that causes a player to lose)? What is the largest number? Justify each answer.

(ii) Bob can always win a game of HT-2 by adopting a particular strategy. Describe the strategy.

For any positive integer n, there is a game HT-n with the same rules as HT-2, except that the game is lost by a player who creates an unbroken sequence of n heads and tails that appears elsewhere in the row. For example, Bob has lost this game of HT-3 by producing a second instance of THT:

$$\begin{array}{cccccccc} a & b & a & b & a & b & a & b \\ H & H & T & T & H & T & H & T \end{array}$$

(iii) Suppose n is odd, and Bob chooses to play by always duplicating Alice's previous play (and Alice knows that this is Bob's strategy). Show that Alice can always win.

In these games, a maximum time of one minute is allowed for each turn.

(iv) Can we be certain that a game of HT-6 will be finished within two hours? Justify your answer.

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2012, Question 7:

Amy and Brian play a game together, as follows. They take it in turns to write down a number from the set $\{0, 1, 2\}$, with Amy playing first. On each turn (except Amy's first turn), the player must not repeat the number just played by the previous player.

In their first version of the game, Brian wins if, after he plays, the sum of all the numbers played so far is a multiple of *3*. For example, Brian will win after the sequence

$$\boxed{2,0}\boxed{1,2}\boxed{1,0}$$

(where we draw a box around each round) because the sum of the numbers is *6*. Amy wins if Brian has not won within five rounds; for example, Amy wins after the sequence

$$\boxed{2,0}\boxed{1,2}\boxed{1,2}\boxed{0,2}\boxed{1,2}$$

(i) Show that if Amy starts by playing either 1 or 2, then Brian can immediately win.

(ii) Suppose, instead, Amy starts by playing 0. Show that Brian can always win within two rounds.

They now decide to change the rules so that Brian wins if, after he plays, the sum of all the numbers played so far is *one less than a multiple of 3*. Again, Amy wins if Brian has not won within five rounds. It is still the case that a player must not repeat the number just played previously.

(iii) Show that if Amy starts by playing either 0 or 2, then Brian can immediately win.

(iv) Suppose, instead, Amy starts by playing 1. Explain why it cannot benefit Brian to play 2, assuming Amy plays with the best strategy.

(v) So suppose Amy starts by playing 1, and Brian then plays 0. How should Amy play next?

(vi) Assuming both play with the best strategies, who will win the game? Explain your answer.

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2013, Question 7:

$\mathbf{AB}$-words are "words" formed from the letters $\mathbf{A}$ and $\mathbf{B}$ according to certain rules. The rules are applied starting with the empty word, containing no letters. The basic rules are:

(1) If the current word is x, then it can be replaced with the word that starts with $\mathbf{A}$, followed by x and ending with $\mathbf{B}$, written $\mathbf{A}x\mathbf{B}$.

(2) If the current word ends with $\mathbf{B}$, the final $\mathbf{B}$ can be removed.

(i) Show how the word $\mathbf{AAAB}$ can be produced.

(ii) Describe precisely all the words that can be produced with these two rules. Justify your answer. You might like to write $\mathbf{A}^i$ for the word containing just i consecutive copies of $\mathbf{A}$, and similarly for $\mathbf{B}$; for example $\mathbf{A}^3\mathbf{B}^2 = \mathbf{AAABB}$.

We now add a third rule:

(3) Reverse the word, and replace every $\mathbf{A}$ by $\mathbf{B}$, and every $\mathbf{B}$ by $\mathbf{A}$.

For example, applying this rule to $\mathbf{AAAB}$ would give $\mathbf{ABBB}$.

(iii) Describe precisely all the words that can be produced with these three rules. Justify your answer.

Finally, we add a fourth rule:

(4) Reverse the word.

(iv) Show that every word consisting of $\mathbf{A}$s and $\mathbf{B}$s can be formed using these four rules. Hint: show how, if we have produced the word w, we can produce (a) the word $\mathbf{A}w$, and (b) the word $\mathbf{B}w$; hence deduce the result.

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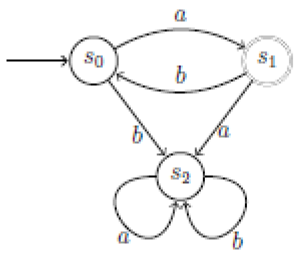
2014, Question 7:

A *finite automaton* is a mathematical model of a simple computing device. A small finite automaton is illustrated below.



A finite automaton has some *finite* number of states; the above automaton has three states, labelled s_0, s_1 and s_2. The initial state, s_0, is indicated with an incoming arrow. The automaton receives *inputs* (e.g. via button presses), which might cause it to change state. In the example, the inputs are a and b. The state changes are illustrated by arrows; for example, if the automaton is in state s_1 and it receives input b, it changes to state s_0; if it is in state s_2 and receives either input, it remains in state s_2. (For each state, there is precisely one out-going arrow for each input.)

Some of the states are defined to be *accepting states*; in the example, just s_1 is defined to be an accepting state, represented by the double circle. A *word* is a sequence of inputs. The automaton *accepts* a word w if that sequence of inputs leads to an accepting state from the initial state. For example, the above automaton accepts the word aba.

(i) Write down a description of the set of words accepted by the above automaton. A clear but informal description will suffice.

(ii) Suppose we alter the above automaton by swapping accepting and non-accepting states; i.e. we make s_0 and s_2 accepting, and make s_1 non-accepting. Write down a description of the set of words accepted by this new automaton. Again, a clear but informal description will suffice.

(iii) Draw an automaton that accepts all words containing an even number (possibly zero) of a's and any number of b's (and no other words).

(iv) Now draw an automaton that accepts all words containing an even number of a's *or* an odd number of b's (and no other words).

Let a^n represent n consecutive a's. Let L be the set of all words of the form a^nb^n where $n = 0, 1, 2, \ldots$; i.e. all words composed of some number of a's followed by the *same* number of b's. We will show that no finite automaton accepts precisely this set of words.