FP3 Series & Limits Questions

2 (a) Find $\int_0^a xe^{-2x}\,dx$, where $a > 0$. *(5 marks)*

(b) Write down the value of $\lim_{a\to\infty} a^k e^{-2a}$, where k is a positive constant. *(1 mark)*

(c) Hence find $\int_0^\infty xe^{-2x}\,dx$. *(2 marks)*

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4 (a) Use the series expansion

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots$$

to write down the first four terms in the expansion, in ascending powers of x, of $\ln(1-x)$. *(1 mark)*

(b) The function f is defined by

$$f(x) = e^{\sin x}$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$1 + x + \tfrac{1}{2}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(c) Find

$$\lim_{x\to 0} \frac{e^{\sin x} - 1 + \ln(1-x)}{x^2 \sin x}$$ *(4 marks)*

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5 (a) Show that $\lim_{a\to\infty}\left(\frac{3a+2}{2a+3}\right) = \frac{3}{2}$. *(2 marks)*

(b) Evaluate $\int_1^\infty \left(\frac{3}{3x+2} - \frac{2}{2x+3}\right)dx$, giving your answer in the form $\ln k$, where k is a rational number. *(5 marks)*

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7 (a) (i) Write down the first three terms of the binomial expansion of $(1+y)^{-1}$, in ascending powers of y. *(1 mark)*

(ii) By using the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

and your answer to part (a)(i), or otherwise, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\sec x$ are

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}$$ *(5 marks)*

(b) By using Maclaurin's theorem, or otherwise, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\tan x$ are

$$x + \frac{x^3}{3}$$ *(3 marks)*

(c) Hence find $\lim_{x \to 0} \left(\frac{x \tan 2x}{\sec x - 1} \right)$. *(4 marks)*

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4 (a) Explain why $\int_0^{e} \frac{\ln x}{\sqrt{x}} \, dx$ is an improper integral. *(1 mark)*

(b) Use integration by parts to find $\int x^{-\frac{1}{2}} \ln x \, dx$. *(3 marks)*

(c) Show that $\int_0^{e} \frac{\ln x}{\sqrt{x}} \, dx$ exists and find its value. *(4 marks)*

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6 The function f is defined by $f(x) = (1+2x)^{\frac{1}{2}}$.

(a) (i) Find $f'''(x)$. *(4 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$f(x) \approx 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$ *(4 marks)*

(b) Use the expansion of e^x together with the result in part (a)(ii) to show that, for small values of x,

$$e^x(1+2x)^{\frac{1}{2}} \approx 1 + 2x + x^2 + kx^3$$

where k is a rational number to be found. *(3 marks)*

(c) Write down the first four terms in the expansion, in ascending powers of x, of e^{2x}.
(1 mark)

(d) Find

$$\lim_{x \to 0} \frac{e^x(1+2x)^{\frac{1}{2}} - e^{2x}}{1 - \cos x}$$ *(4 marks)*

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6 (a) The function f is defined by

$$f(x) = \ln(1 + e^x)$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$\ln 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(b) Hence write down the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\frac{1+e^x}{2}\right)$. *(1 mark)*

(c) Use the series expansion

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

to write down the first three terms in the expansion, in ascending powers of x, of $\ln\left(1-\frac{x}{2}\right)$. *(1 mark)*

(d) Use your answers to parts (b) and (c) to find

$$\lim_{x \to 0} \left[\frac{\ln\left(\frac{1+e^x}{2}\right) + \ln\left(1-\frac{x}{2}\right)}{x - \sin x} \right]$$ *(4 marks)*

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7 (a) Write down the value of

$$\lim_{x \to \infty} xe^{-x}$$ *(1 mark)*

(b) Use the substitution $u = xe^{-x} + 1$ to find $\int \frac{e^{-x}(1-x)}{xe^{-x}+1}\, dx$. *(2 marks)*

(c) Hence evaluate $\int_1^{\infty} \frac{1-x}{x+e^x}\, dx$, showing the limiting process used. *(4 marks)*