√2 is Irrational

Appendix 2. Euclid's Proof that √2 is Irrational

Euclid's aim was to prove that √2 could not be written as a [illegible] Because he was using proof by contradiction, the first step was to [illegible] that the opposite was true, that is to say, that √2 could be written as [illegible] unknown fraction. This hypothetical fraction is represented by [illegible] *p* and *q* are two whole numbers.

Before embarking on the proof itself, all that is required is [illegible] understanding of some properties of fractions and even numbers.

(1) If you take any number and multiply it by 2, then the new [illegible] must be even. This is virtually the definition of an even number.

(2) If you know that the square of a number is even, then the number itself must also be even.

(3) Finally, fractions can be simplified: $\frac{16}{24}$ is the same as $\frac{8}{12}$; just divide the top and bottom of $\frac{16}{24}$ by the common factor 2. Furthermore, $\frac{8}{12}$ is the same as $\frac{4}{6}$, and in turn $\frac{4}{6}$ is the same as $\frac{2}{3}$. However, $\frac{2}{3}$, cannot be simplified any further because 2 and 3 have no common factors. It is impossible to keep on simplifying a fraction forever.

Now, remember that Euclid believes that √2 cannot be written as a fraction. However, because he adopts the method of proof by contradiction, he works on the assumption that the fraction p/q does exist and then he explores the consequences of its existence:

$$\sqrt{2} = p/q.$$

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If we square both sides, then

$$2 = p^2/q^2.$$

This equation can easily be rearranged to give

$$2q^2 = p^2.$$

Now from point (1) we know that p^2 must be even. Furthermore, from point (2) we know p itself must also be even. But if p is even, then it can be written as $2m$, where m is some other whole number. This follows from point (1). Plug this back into the equation and we get

$$2q^2 = (2m)^2 = 4m^2.$$

Divide both sides by 2, and we get

$$q^2 = 2m^2.$$

But by the same arguments we used before, we know that q^2 must be even, and so q itself must also be even. If this is the case, then q can be written as $2n$, where n is some other whole number. If we go back to the beginning, then

$$\sqrt{2} = p/q = 2m/2n.$$

The $2m/2n$ can be simplified by dividing top and bottom by 2, and we get

$$\sqrt{2} = m/n.$$

We now have a fraction m/n, which is simpler than p/q.

However, we now find ourselves in a position whereby we can exactly the same process on m/n, and at the end of it we will generate even simpler fraction, say g/h. This fraction can then be put through mill again, and the new fraction, say e/f, will be simpler still. We can this through the mill again, and repeat the process over and over with no end. But we know from point (3) that fractions cannot simplified forever. There must always be a simplest fraction, but original hypothetical fraction p/q does not seem to obey this Therefore, we can justifiably say that we have reached a contradic- $\sqrt{2}$ could be written as a fraction the consequence would be absurd so it is true to say that $\sqrt{2}$ cannot be written as a fraction. Therefore an irrational number.

$\sqrt{2}$ is Irrational

The argument is in fact a classic example of proof by contradiction. We begin, in other words, by supposing that $\sqrt{2}$ *can* be written as a fraction. Then, by reducing that fraction to its 'lowest terms', i.e. by cancelling out any common factors, we obtain $\sqrt{2} = m/n$, where m and n are whole numbers which have no common factor.

To see a contradiction develop, begin by squaring both sides to obtain $2 = m^2/n^2$, so that $m^2 = 2n^2$. This means that m^2 is twice a whole number, so m^2 is even. It follows that *m must be even* (for if m were odd, m^2 would be odd, as odd $\times$ odd $=$ odd).

Now , as m is even, it can be written as $2r$, where r is a whole number. The equation $m^2 = 2n^2$ can then be rewritten as $4r^2 = 2n^2$, i.e. $n^2 = 2r^2$. So n^2 is even, and by the same argument as before, *n must be even*.

And there is the contradiction: m and n started by having no common factor, yet must now have a common factor of 2, because they are both even.

The only way out of this absurd situation is for the original assumption – that $\sqrt{2}$ can be written as a ratio of two whole numbers – to be false.

So $\sqrt{2}$ is an irrational number. And there are plenty of others; there is nothing exceptional or peculiar about them. In fact, there are 'more' irrational numbers than there are rational ones, though what this statement means, exactly, takes a bit of thinking about, as the two things we are comparing are both infinite.

Infinitely Many Primes

For a deeper example of proof by contradiction we turn to the subject of *prime numbers*.

Now, a prime number is a whole number, larger than 1, which is divisible only by 1 and itself. So

$$2, 3, 5, 7, 11, 13, 17, 19 \ldots$$

are all prime, but 15, for example, is not, because it is divisible by 3 and by 5.

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541

The first 100 prime numbers.

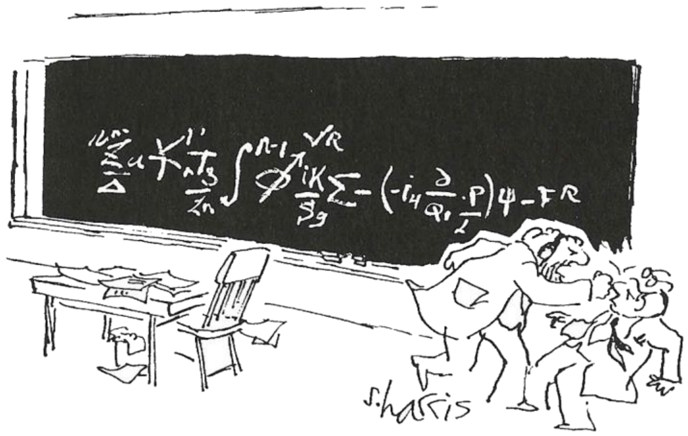
Every whole number greater than 1 is either prime or can be written as a product of primes. So, for instance, 17 is prime but 18 can be written as $2 \times 3 \times 3$. In this sense, primes are the 'building blocks' out of which other whole numbers can be created by multiplication.

As we proceed up the list of whole numbers, primes occur quite frequently at first, but less frequently later on. Thus 25% of the numbers up to 100 are prime, but the corresponding figure for numbers up to 1,000,000 is just 7.9%.

An obvious question, then, is: does the list of primes come to a complete stop somewhere, or does it go on for ever?

And Euclid discovered the answer: *there are infinitely many prime numbers.*

How, then, did he *prove* it?



"YOU WANT PROOF? I'LL GIVE YOU PROOF!"

The answer is that he turned to proof by contradiction.

He began, then, by supposing that the number of primes is finite, in which case there will be some *largest* prime number, which we will call p. The complete list of primes will then be

$$2, 3, 5, 7, 11, 13, \ldots, p.$$

So far, so good. Even straightforward, you might say. But the next step is an inspired one.

Euclid's ingenious idea was to consider the number

$$N = 2 \times 3 \times 5 \times \ldots \times p + 1$$

i.e. the number obtained by multiplying all the primes together and adding 1.

Now, this number is certainly greater than p, and as p is the largest prime this new number N cannot be prime. It must therefore be possible to write it as a product of primes, i.e. it must be divisible by at least one prime number.

But it isn't; if you divide N by any prime number from the list $2, 3, 5, \ldots, p$ you always get a remainder of 1.

We have arrived at a contradiction, then, and the only way out is for the original hypothesis to be wrong; the number of primes cannot be finite – it must be infinite.

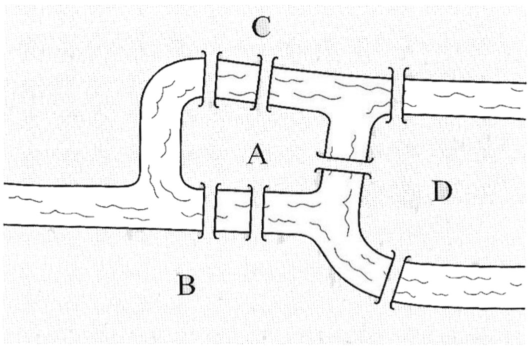
Bridges of Konigsberg

The basic idea is to prove that some proposition is true by exploring the possibility that it is false, and then showing that this would lead to a contradiction or nonsense of some kind. So the proposition can't be false, and the only possibility then left is that it is true.

This whole line of argument is sometimes called the 'indirect' method of proof, or *reductio ad absurdum*.

As a first example, consider the so-called *Königsberg Bridge Problem*, which came to the attention of the great Swiss mathematician Leonhard Euler in 1736.

At the time, Königsberg was a town in East Prussia, divided by the River Pregel into several parts which were connected by seven bridges.



The citizens of Königsberg crossed these bridges on their long, leisurely Sunday afternoon walks. And they were vexed – so the story goes – by one particular question: can you take a walk in Königsberg in such a way that you cross each of the seven bridges *once and only once*?

Now, at first sight we are faced with the tedious and daunting task of considering all the possible routes in turn, and showing that none of them works. But, as Euler showed, there is a clever way of circumventing all this. And one convincing way of presenting the argument is as a proof by contradiction.

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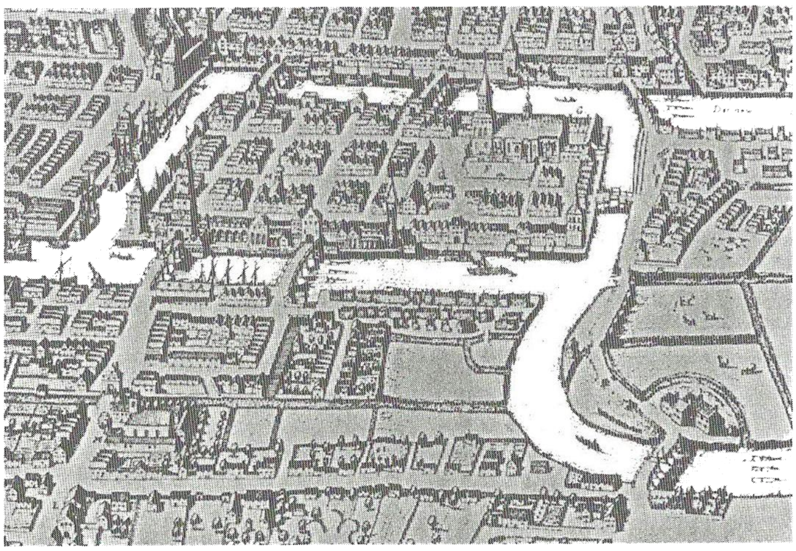
Suppose, then, that it *is* possible. We start, in other words, in one of the four regions A, B, C, D and end up at one of them (possibly the same one), having crossed each of the seven bridges exactly once.

Now, it follows immediately that there will be at least two regions which are neither at the beginning nor at the end of the walk. Consider one of these regions. We visit it a certain number of times and leave it an equal number of times, and as we cross each bridge exactly once it follows that there must be an even number of bridges leading from this region.

But no region in the Königsberg figure above has this property: the island A has 5 bridges leading from it, while the regions B, C and D all have 3 each.

So you *can't* take a walk in Königsberg in this particular way.



At least, you couldn't in 1736. As I understand it, the situation has now changed; for Königsberg is now Kaliningrad, and has only five bridges, most of them the result of rebuilding after the Second World War.