A Level Questions of a Problem-Solving Variety

1. Prove from first principles that
   a. If $y = 3x^2 + 2x$, then $\frac{dy}{dx} = 6x + 2$
   b. If $g(x) = \cos(x)$, then $g'(x) = -\sin(x)$

2. A colony of ants grows exponentially according to the formula
$$P = Ae^{kt}$$
Where P is the number of ants in the colony (measured in thousands), t is time measured in days and A, k are both constants. Initially, the colony of ants numbers 1000 and after 10 days the population has risen to 1500.

   a. Compute the population of ants in the colony after 15 days.

   In a different colony of ants, a graph of $\log P$ against t is plotted resulting in a straight line. The gradient of the line drawn is 0.012 and it is known that the initial population of ants is equal to that in the previous population.

   b. Find the difference between the number of ants in both colonies after 10 days.
   c. Calculate the total number of days required for this colony of ants to first exceed the number of ants after 10 days in the first colony.

3. Equivalent expressions when using logarithms are $a^x = b \iff x = \log_a b$. By considering a logarithm of base c $(c \neq a)$,
   a. Show that $\log_a b = \frac{\log_c b}{\log_c a}$.

   Hence, find the exact value of

   b. $\frac{\log_4 8}{\log_4 2}$.

   And show that

   c. $\frac{\log_6 24}{\log_6 12} = 1 + \log_{12} 2$.

   d. Without using a calculator, explain why $1 + \log_{12}\left(\frac{1}{2}\right) < 1$.

4. The position of a particle is modelled by the parametric equations
$$x = 1 + 2\sin t \quad \text{and} \quad y = 1 + 2\cos 2t$$
Where $0 \leq t \leq 2\pi$. The particle traces out a trajectory, C, from $t = 0$ to $t = 2\pi$.

   a. Show that the trajectory C satisfies the equation $y = 3 - (x - 1)^2$
   b. By considering the domain and range of C, sketch the curve C, clearly indicating the minimum point.

c. Find an expression for the gradient of the trajectory and deduce that for $0 \leq t \leq \pi$, the gradient function is a decreasing function. (You do not need to be concerned about the situation when $t \neq \frac{\pi}{2}$).

5. Explain why the choice of $x = \frac{1}{2}$ is valid in the Binomial Expansion of $f(x)$, where
$$f(x) = \frac{1 + 2x}{\sqrt[3]{8 - 4x}}$$
Hence, by computing the first three terms of the binomial expansion of $f(x)$, find an approximation for $\sqrt[3]{\frac{4}{3}}$.

6. Without using differentiation, find the maximum and minimum values of $g(x)$, where
$$g(x) = 1 + 2\cos x + 3\sin x \qquad \{x : x \in \mathbb{R}, x > 0\}$$
and find the first value of x for which a maximum and a minimum occur, correct to one decimal place.

7.
   a. By first finding the partial fractions within the integral, find, giving your answer as a natural logarithm
$$\int_0^1 \frac{x+9}{9-x^2}\,dx$$
   b. Find the value of
$$\int_0^1 \frac{9-x^2}{x+9}\,dx$$
   Giving your answer in an exact form.

8. Prove the Newton-Raphson formula for finding an approximate solution to the equation $f(x) = 0$.

9. In this question, you may assume that there is only one root of $f(x)$. Using the Newton-Raphson formula, find an approximation to the solution of $f(x) = \sin(x^2) + x - 1$.

10. Money is invested in a bank account that pays 1.2% interest per annum. Initially, I invest £1,000. The total amount in the account after n years is given by T_n, where T_n is a geometric sequence.
    a. Show that $T_n = 1000 \times (1.012)^{n-1}$
    b. Find how many years I will have to wait for the amount in the account to first exceed £1,300.

11. By first writing the expression $y = a^x$ in the form of an exponential with base e, show that
    a. $\frac{dy}{dx} = a^x \ln a$
    b. $\int a^x dx = \frac{1}{\ln a} a^x + c$
    c. Prove the result in a. again, but this time taking natural logarithms first.

d. Hence, find an expression for the exact value of x for which the curve $y = 3^x$ takes has a gradient of $\ln 2$.

12. Let $k \in \mathbb{Z}^+$ and

$$I = \int_k^{2k} \frac{1}{(2x+1)^2} dx$$

a. Explain why
   i. $I < 0$ for all values of k.
   ii. $I \to 0$ as $k \to \infty$
b. How are your answers to a. affected if instead, $k \in \mathbb{Z}^-$?

13. The vector $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\overrightarrow{OB} = -4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. The vector $\overrightarrow{OC}$ has the same magnitude and is parallel to the vector $\overrightarrow{AB}$, whilst the position vector of the point D is parallel to $\overrightarrow{BC}$ and $|\overrightarrow{OD}|: |\overrightarrow{BC}| = 1: 2$.

Find the vector $\overrightarrow{DA}$ and $\angle DOB$.

14. A conical tank of radius $r = 1$m and height $h = 2$m is initially full. At time $t = 0$, the tank begins to link from three holes: the first leaks at a rate of 10% the current volume, the second at a rate of 20% the current volume and the third at a rate of 30% the current volume. To stop the tank from running empty, liquid is added at a rate of 1m^3/s.

a. Explain why, for all times, $h = 2r$.
b. As time progresses, the level of water in the tank settles to a volume V_T, where the input of 1m^3/s is balanced by the loss of liquid from the tank. Find the value of V_T;
   i. Without solving the differential equation at all.
   ii. By first solving the differential equation.
c. Show further that

$$2\pi r^2 \frac{dr}{dt} = 1 - 0.4\pi r^3$$

and **by solving the differential equation above**, find the radius of the liquid in the tank and hence the height of liquid in the tank as the volume approaches its limiting value.
d. Can you find the same result by not solving the differential equation above?

15. Baby weight, w, is conjected to increase logarithmically with time, t. The population of baby weights are grouped into percentiles, with a baby on the 91st percentile for their weight meaning that they are among the top 9% of baby weights nationally. Immediately after a birth, the weight of a baby fluctuates and settles down usually after a couple of weeks.

For a baby tracking the 91st percentile after 2 weeks, they would weigh 4.5kg and after 52 weeks they would weigh 11.1kg. After 6 months, a baby following this weight gain line would be expected to weigh around 9.2kg.

To test the conjecture, a model of the form

$$t = Ae^{kw}$$

Is proposed.

a. Why could such a model be proposed when a 'logarithmic increase' is stated?

b. Comment on the accuracy of the model by comparing the model at 6 months against the real-life data.

16. A logistic equation is defined by the rate of change of P being proportional to the product of P with $1-P$. In this logistic equation, for all time, $0 < P < 1$ and assume that P is a continuous variable.

    Suppose that at $t=0, P=P_0$ and that at time $t=1, P=\frac{P_0}{3}$, where $0 < P_0 < 1$ is a constant. Assume further that the model is valid only for $0 < t < 1$.

    Show that the constant of proportionality is

$$k = \ln\left|\frac{1-P_0}{3-P_0}\right|$$

    and deduce in the case of $P_0 = \frac{1}{2}$, that $\frac{dp}{dt} < 0$ for the specified domain.

17. Recall that for small values of x, $\cos x \approx 1 - \frac{x^2}{2}$. By considering the first three non-zero terms of a binomial expansion and picking a suitable value of x, find an approximate value for $\sqrt{\sec(0.8)}$, correct to three decimal places. Comment on the accuracy of your result by comparing the approximate value against the true value.

18. Find the greatest possible domain of the function $f(x) = \sqrt{1-4x+x^2}$. For the domain $x \geq 5+\sqrt{3}$, find the range of the function $g(x) = \frac{1}{f(x)}$.

19. Let

$$x = 2t - t^2 \qquad \text{and} \qquad y = 4 + t$$

    Where $0 < t < 2$, be parametric equations of the curve $y = f(x)$.

    a. Give a reason why the range of the function $f(x)$ is $4 < f(x) < 6$.
    b. Show that the domain of $f(x)$ is $0 < x \leq 1$.
    c. Find the values of t for which $f(x)$ is an increasing function. Find also the values of t for which $f(x)$ is a decreasing function.
    d. Using parts a. to c., sketch the curve $y = f(x)$.
    e. Show that $x = 10y - y^2 - 24$ and hence deduce a form for the curve $y = f(x)$. What type of function is $f(x)$

20. Let $f(x) = \frac{1}{x^2+2}$ and $g(x) = \frac{x-2}{x+1}$. By considering the partial fractions of $gf(x)$, find an expression for the integral $\int gf(x)\, dx$.
    (**Hint**: you will, at some point, need to use the fact that $\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$; see the formula book)

21. The radiation received on Earth from a Pulsar has Intensity I and for $0 \leq t \leq T$ is modelled by

$$I = 4 + 2.8\sin\omega t$$

Where ω is a constant.

a. State the values of the maximum and minimum intensity of the radiation received on Earth and find, in terms of ω, the times at which the first two maxima and minima occur.

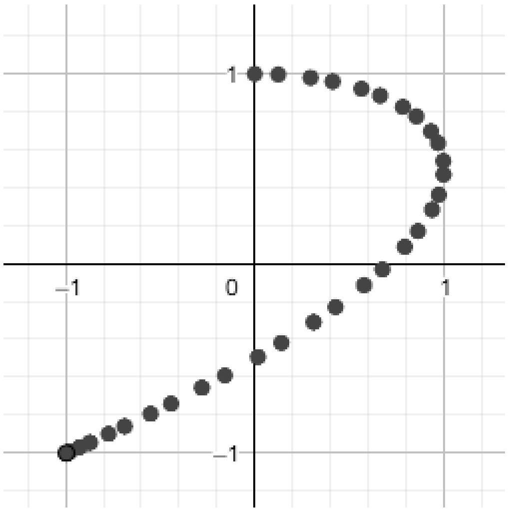
b. Given also that $I = I_0$ when $t = 0$ and when $t = T$ and that exactly one maximum and minimum occur during the time interval $0 \leq t \leq T$, explain why $\omega = \frac{2\pi}{T}$.

22. The points $A(3, 2, 4)$, $B(1,0,1)$ and $C(2, 4, -3)$ are the coordinates of the vertices of the triangle ABC in three dimensions.
Find both the perimeter and area of the triangle ABC.

23. The position of a particle is modelled by the parametric equations

$$x = \sin(3t) \quad \text{and} \quad y = \cos(2t)$$

Where t is time in seconds and $0 < t \leq \pi$. The trajectory of the curve is given by C, an expression of the form $y = f(x)$. The diagram below shows the position of the particle for various times t throughout the specified domain.



Find the gradient of the curve at the times when x is zero and comment on their value. Is this implied by the graph of C above?

Find an expression for the trajectory of the curve C for those times which the trajectory is a decreasing function.

24. Let $f(x) = x^2, x \in \mathbb{R}$ and $g(x) = \frac{x}{1-x}, x \neq 1$ and $x \in \mathbb{R}$.

a. Solve $fg(x) = g^{-1}f(x)$

b. Consider the equation

$$g^{-1}f(x) = x^5$$

i. Show that $x = 0$ is a root.

ii. Show that a non-zero root, α, to the equation exists between 0 and 1.

iii. Show that a Newton-Raphson formula to find an approximate value of the root is given by

$$x_{n+1} = \frac{4x_n^5 + 2x_n^3 + 1}{5x_n^4 + 3x_n^2}$$

and hence find, using a suitable first approximation to the root, the value of α correct to three significant figures.

25. The parabola defined by $y = f(x)$ passes through the points $(0 , 0)$, $(1 , 4)$ and $(2 , 5)$. The point A lies on the curve and has coordinates $(3, a)$. Find the exact value of a.

26. A parabola is symmetric about the line $x = 2$ and has a repeated root. The parabola also passes through the point $(1 , 3)$. Find the exact value of the y-intercept of the parabola.

27. A circle has centre $(4 , 2)$ and passes through the points $(7 , -2)$. The tangent to the circle at the point B is parallel to the tangent at the point $(7 , -2)$. Find the equation of the tangent at the point B.

28. The function $f(x) = x^2 - x + k$, where $k \in \mathbb{R}$ has domain $0 < x < 2$. Prove that the range of $f(x)$ is $k - \frac{1}{4} \leq f(x) \leq k + 2$ and hence find two possible domains in $0 < x < 2$ for which f is a one-to-one function.

29. Let $g(x) = \sqrt{1 - 8x - x^2}$. Find the range of g. Find the domain of the inverse function, if the domain is restricted to $1 \leq x \leq -4 + \sqrt{17}$.

30. Prove that there are infinitely many prime numbers.

31. Prove that $\sqrt{2}$ is irrational.

32. Prove that $\sqrt{5}$ is irrational.

33. Let $x = \log_2(\log_{\frac{1}{2}} \frac{1}{64})$. Show that $x = 1 + \log_2 3$.

34. A cylinder of height $h = 1\text{m}$ is to be produced so that when 2m^3/s of liquid is added to the cylinder and 80% of the volume is lost at that given instant in time, the cylinder remains perfectly full. Assume that the cylinder is initially empty. Show that $2\pi r \frac{dr}{dt} = 2 - \pi r^2 h$ and hence find the radius, correct to one decimal place, of the cylinder.

35.

   a. Find the value of

$$\int_0^1 \frac{5}{(x+2)(x+3)}\, dx$$

   b. Find the value of

$$\int_0^1 \frac{a+b}{(x+a)(x+b)}\, dx$$

   And by picking suitable values of a and b, show that your answer to part a. matches with the answer in part b.

36. Let $x = \sin(t)$ and $y = \cos^2(t)$ for $0 \le t \le \frac{\pi}{2}$. Using parametric integration, find the value of

$$\int y \, dx$$

For the values of t for which x and y are defined.

37. Let

$$\frac{d^2y}{dx^2} = \ln x$$

and suppose that the curve $y = f(x)$ satisfies the conditions that $y = \frac{dy}{dx} = 1$ when $x = 1$. Find the exact value of y when $x = 2$.

38. Let $N = ae^{bt} + c$, where a, b and c are constants. Show that

$$\frac{dN}{dt} = b(N - c)$$

And hence deduce that

$$\frac{d^4N}{dt^4} = b^4(N - c)$$

39. The function $f(x) = 2^x - x^2$ has exactly two stationary points, and let α and β be the x-coordinates of those stationary points. Let $\alpha < \beta$. Show that $0 < \alpha < 1$, that $3 < \beta < 4$ and by using Newton-Raphson iteration, find the values of α and β, correct to two decimal places.