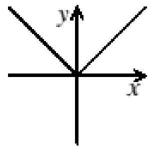
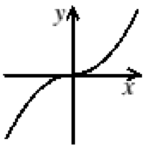
UKMT Equations Questions
(Answers follow after all the questions)

2005…

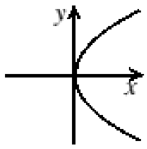
19. The graph of $y = |x|$ is shown alongside.
Which of the following could be a sketch of the graph of $y = x\,|x|$?



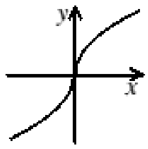
A



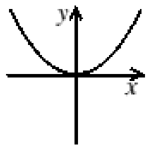
B



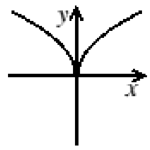
C



D



E



2006…

2. If $6x - y = 21$ and $6y - x = 14$, what is the value of $x - y$?

A 1 B 2 C 3 D 4 E 5

17. A trapezium is bounded by four lines, the equations of which are $x = 0$, $x = 4$, $4y = 3x + 8$ and $y = k$, where $k < 2$.
For which value of k is the numerical value of the perimeter of the trapezium equal to the numerical value of the area of the trapezium?

A $\frac{3}{2}$ B 1 C $\frac{1}{2}$ D $-\frac{1}{2}$ E -1

2007…

25. The line with equation $y = x$ is an axis of symmetry of the curve with equation $y = \dfrac{px + q}{rx + s}$, where p, q, r, s are all non-zero. Which of the following is necessarily true?

A $p + q = 0$ B $r + s = 0$ C $p + r = 0$ D $p + s = 0$ E $q + r = 0$

2008…

16. The numbers x, y and z satisfy the equations

$$x + y + 2z = 850, \qquad x + 2y + z = 950, \qquad 2x + y + z = 1200.$$

What is their mean?

A 250 B $\dfrac{1000}{3}$ C 750 D 1000 E More information is needed.

23. How many pairs of real numbers (x, y) satisfy the equation $(x + y)^2 = (x + 3)(y - 3)$?

A 0 B 1 C 2 D 4 E infinitely many

2009…

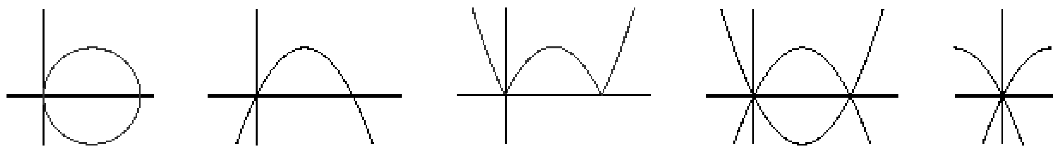
13. Suppose that $x - \frac{1}{x} = y - \frac{1}{y}$ and $x \neq y$. What is the value of xy?

A 4 B 1 C −1 D −4 E more information is needed

16. The positive numbers x and y satisfy the equations $x^4 - y^4 = 2009$ and $x^2 + y^2 = 49$. What is the value of y?

A 1 B 2 C 3 D 4 E more information is needed

18. Which of the following could be part of the graph of the curve $y^2 = x(2 - x)$?



A B C D E

2010…

8. Which of the following is equivalent to $(x + y + z)(x - y - z)$?

A $x^2 - y^2 - z^2$ B $x^2 - y^2 + z^2$ C
$x^2 - xy - xz - z^2$
D $x^2 - (y + z)^2$ E $x^2 - (y - z)^2$

9. The symbol ◊ is defined by $x \lozenge y = x^y - y^x$. What is the value of $(2 \lozenge 3) \lozenge 4$?

A −3 B $-\frac{3}{4}$ C 0 D $\frac{3}{4}$ E 3

22. If $x^2 - px - q = 0$, where p and q are positive integers, which of the following could not equal x^3?

A $4x + 3$ B $8x + 5$ C $8x + 7$ D $10x + 3$ E $26x + 5$

2011…

6. Granny and her granddaughter Gill both had their birthday yesterday. Today, Granny's age in years is an even number and 15 times that of Gill. In 4 years' time Granny's age in years will be the square of Gill's age in years. How many years older than Gill is Granny today?

A 42 B 49 C 56 D 60 E 64

15. The equation $x^2 + ax + b = 0$, where a and b are different, has solutions $x = a$ and $x = b$. How many such equations are there?

A 0 B 1 C 3 D 4 E an infinity

18. Two numbers x and y are such that $x + y = 20$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$. What is the value of $x^2y + xy^2$?

A 80 B 200 C 400 D 640 E 800

20. Positive integers x and y satisfy the equation $\sqrt{x} - \sqrt{11} = \sqrt{y}$.

What is the maximum possible value of $\frac{x}{y}$?

A 2 B 4 C 8 D 11 E 44

23. What is the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$?

A -7 B -5 C -4 D -1 E 4

2012…

7. Given that $x + y + z = 1$, $x + y - z = 2$ and $x - y - z = 3$, what is the value of xyz?

A -2 B $-\frac{1}{2}$ C 0 D $\frac{1}{2}$ E 2

25. How many distinct pairs (x, y) of real numbers satisfy the equation $(x+y)^2 = (x+4)(y-4)$?

A 0 B 1 C 2 D 3 E 4

2013…

5. The numbers x and y satisfy the equations $x(y + 2) = 100$ and $y(x + 2) = 60$. What is the value of $x - y$?

A 60 B 50 C 40 D 30 E 20

2014…

14. Given that $\dfrac{3x + y}{x - 3y} = -1$, what is the value of $\dfrac{x + 3y}{3x - y}$?

A -1 B 2 C 4 D 5 E 7

2015…

2. What is the sum of all the solutions of the equation $6x = \dfrac{150}{x}$?

A 0 B 5 C 6 D 25 E 156

25. A function, defined on the set of positive integers, is such that $f(xy) = f(x) + f(y)$ for all x and y. It is known that $f(10) = 14$ and $f(40) = 20$. What is the value of $f(500)$?

A 29 B 30 C 39 D 48 E 50

2016…

9. A square has vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. Graphs of the following equations are drawn on the same set of axes as the square.

$$x^2 + y^2 = 1, \qquad y = x + 1, \qquad y = -x^2 + 1, \qquad y = x, \qquad y = \frac{1}{x}$$

How many of the graphs pass through exactly two of the vertices of the square?

A 1 B 2 C 3 D 4 E 5

UKMT Equations Answers

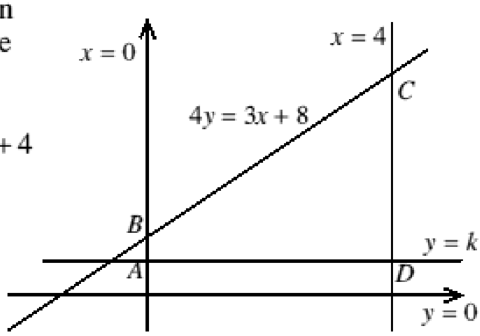
2005...

19. **A** When $x \geqslant 0$, $|x| = x$, so $x|x| = x^2$; when $x < 0$, $|x| = -x$, so $x|x| = -x^2$. Only graph A has the same shape as the graph of $y = x^2$ for $x \geqslant 0$ and the same shape as the graph of $y = -x^2$ for $x < 0$.

2006...

2. **A** Subtracting the second equation from the first: $6x - y - (6y - x) = 21 - 14$. So $7x - 7y = 7$, that is $x - y = 1$.
(The equations may be solved to give $x = 4$, $y = 3$, but it is not necessary to do this in order to find the value of $x - y$.)

17. **E** The trapezium in question is shown as $ABCD$ in the diagram. The coordinates of its vertices are $A(0, k)$, $B(0, 2)$, $C(4, 5)$, $D(4, k)$. Using Pythagoras' Theorem: $BC = \sqrt{4^2 + 3^2} = 5$.
The perimeter of $ABCD = (2 - k) + 5 + (5 - k) + 4$
$= 16 - 2k$.
The area of $ABCD = 4(2 - k) + \frac{1}{2} \times 4 \times 3$
$= 14 - 4k$. So $16 - 2k = 14 - 4k$, that is $k = -1$.
(In the diagram it was assumed that $k > 0$, although it transpires that $k < 0$. However, this does not affect the validity of the solution.)



2007...

25. **D** As the line $y = x$ is an axis of symmetry of the curve, if the point (a, b) lies on the curve, so too does the point (b, a). Hence the equation of the curve may also be written as
$x = \dfrac{py + q}{ry + s}$.
Therefore, substituting for x in the original equation:

$$y = \frac{p\left(\dfrac{py + q}{ry + s}\right) + q}{r\left(\dfrac{py + q}{ry + s}\right) + s} = \frac{p(py + q) + q(ry + s)}{r(py + q) + s(ry + s)}.$$

Therefore $$y\big(r(py + q) + s(ry + s)\big) = p(py + q) + q(ry + s),$$

that is $y^2r(p + s) + y\left(qr + s^2 - p^2 - qr\right) - pq - qs = 0,$

that is $$(p + s)\left(y^2r + y(s - p) - q\right) = 0.$$

Since r is non-zero, the expression in the second bracket is non-zero for all but at most two values of y. Hence $p + s = 0$.

2008…

16. A Adding the three given equations gives $4(x + y + z) = 3000$. Therefore $x + y + z = 750$. So the mean is $\frac{750}{3} = 250$.

23. B Let $X = x + 3$ and $Y = y - 3$. Then the given equation becomes $(X + Y)^2 = XY$. So $X^2 + XY + Y^2 = 0$. However X^2, Y^2 and $XY (= (X + Y)^2)$ are non-negative. Hence $X = Y = 0$; so $x = -3$ and $y = 3$ is the only solution.

2009…

13. C $x - \frac{1}{x} = y - \frac{1}{y}$ hence $x^2y - y = xy^2 - x$. Thus $xy(y - x) + y - x = 0$.
Therefore $(y - x)(xy + 1) = 0$. As $x \neq y$ then $y - x \neq 0$.
Hence $xy + 1 = 0$ giving $xy = -1$.

16. B Since $x^4 - y^4 = 2009$ it follows that $(x^2 + y^2)(x^2 - y^2) = 2009$.
But $x^2 + y^2 = 49$ hence $x^2 - y^2 = \frac{2009}{49} = 41$.
Subtracting gives $2y^2 = 8$ hence $y^2 = 4$. Since $y > 0$, $y = 2$.

18. A We have $y^2 = x(2 - x)$. Now $y^2 \geqslant 0$ for all real y hence $x(2 - x) \geqslant 0$.
Hence $0 \leqslant x \leqslant 2$. In fact we can rewrite the equation as $(x - 1)^2 + y^2 = 1$; so this is a circle of radius 1 with centre (1,0).

2010…

8. D $(x + y + z)(x - y - z) = [x + (y + z)][x - (y + z)] = x^2 - (y + z)^2$.

9. D $(2 \diamond 3) \diamond 4 = (2^3 - 3^2) \diamond 4 = (-1) \diamond 4 = (-1)^4 - 4^{-1} = 1 - \frac{1}{4} = \frac{3}{4}$.

22. B Since $x^2 - px - q = 0$, it follows that $x^3 = px^2 + qx$.
But $x^2 = px + q$ and so $x^3 = p(px + q) + qx$, ie $x^3 = (p^2 + q)x + pq$.
The three possible values shown for pq are 3, 5 and 7.
If $pq = 3$, $p^2 + q = 1^2 + 3 = 4$ or $p^2 + q = 3^2 + 1 = 10$. Hence $4x + 3$ and $10x + 3$ could equal x^3.
If $pq = 7$, we may take $p = 1, q = 7$ to get $p^2 + q = 1^2 + 7 = 8$. Hence $8x + 7$ could equal x^3.
If $pq = 5$, we may take $p = 5, q = 1$ to get $p^2 + q = 5^2 + 1 = 26$. Hence $26x + 5$ could equal x^3.
However, the only other possibility, $p = 1, q = 5$ gives $p^2 + q = 6 \neq 8$. Therefore $8x + 5 \neq x^3$.

2011…

6. **C** Let Granny's age today be G and Gill's age today be g.
Therefore $G = 15g \ldots (1)$ and $G + 4 = (g + 4)^2 \ldots (2)$.
Substituting (1) into (2) gives $15g + 4 = g^2 + 8g + 16$, hence $g^2 - 7g + 12 = 0$.
Thus $(g - 3)(g - 4) = 0$, hence $g = 3$ or 4.
As G is even and $G = 15g$, g is also even. Thus $g = 4$ and $G = 15 \times 4 = 60$.
Hence today, Granny is 56 years older than Gill.

15. **B** If a, b are roots of $x^2 + ax + b = 0$ then $x^2 + ax + b = 0$ must be $(x - a)(x - b) = 0$. As $(x - a)(x - b) = x^2 + (-a - b)x + ab$ then $a = -a - b$ and $b = ab$. If $b = 0$ we see immediately that $a = 0$. But this is not possible as a and b are different. If $b \neq 0$ then $a = 1$ and $b = -2$. So there is just one solution pair.

18. **E** Multiplying $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ throughout by $2xy$ gives $2y + 2x = xy$. Hence $xy = 2(x + y) \ldots (1)$.
But since $x^2y + xy^2 = xy(x + y)$, we can use (1) to give $xy(x + y) = 2(x + y)(x + y)$.
But $x + y = 20$, hence $x^2y + xy^2 = 2 \times 20^2 = 800$.

20. **B** Squaring the equation $\sqrt{x} - \sqrt{11} = \sqrt{y}$ gives $x - 2\sqrt{11x} + 11 = y \ldots (1)$. You see here that $2\sqrt{11x}$ is an integer. Thus $x = 11a^2$ for some integer a. Hence in (1), $y = 11a^2 - 22a + 11 = 11(a^2 - 2a + 1)$. Thus $\frac{x}{y} = \left(\frac{a}{a - 1}\right)^2$ whose maximum value, for integer a, is easily seen to be $\left(\frac{2}{1}\right)^2 = 4$.

23. **B** $x^2 + y^2 + 2xy + 6x + 6y + 4 = (x + y)^2 + 6(x + y) + 4 = [(x + y) + 3][(x + y) + 3] - 5 = (x + y + 3)^2 - 5$. But $(x + y + 3)^2 \geqslant 0$ for all values of x and y. As $x + y + 3$ can be 0 for appropriate values of x, y the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$ is -5.

2012…

7. **D** Add the first and third equations: $2x = 4$, so $x = 2$. Add the first two equations: $2x + 2y = 3$, so $y = -\frac{1}{2}$. Substitute for x and y in the first equation: $2 + \left(-\frac{1}{2}\right) + z = 1$ so $z = -\frac{1}{2}$. Therefore $xyz = 2 \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) = \frac{1}{2}$.

25. **B** Starting with $(x + y)^2 = (x + 4)(y - 4)$ and expanding both sides gives $x^2 + 2xy + y^2 = xy - 4x + 4y - 16$, i.e. $x^2 + (y + 4)x + y^2 - 4y + 16 = 0$.
To eliminate the xy term we let $z = x + \frac{1}{2}y$ and then replace x by $z - \frac{1}{2}y$. The equation above becomes $z^2 + 4\left(z - \frac{1}{2}y\right) + \frac{3}{4}y^2 - 4y + 16 = 0$. However,

$$z^2 + 4\left(z - \tfrac{1}{2}y\right) + \tfrac{3}{4}y^2 - 4y + 16 = (z + 2)^2 + \tfrac{3}{4}y^2 - 6y + 12$$

$$= (z + 2)^2 + \tfrac{3}{4}\left(y^2 - 8y + 16\right) = (z + 2)^2 + \tfrac{3}{4}(y + 4)^2.$$

So the only real solution is when $z = -2$ and $y = 4$; i.e. $x = -4$ and $y = 4$.

2013…

5. **E** Since $x(y + 2) = 100$ and $y(x + 2) = 60$ then $xy + 2x = 100$ and $xy + 2y = 60$.
Subtracting gives $2x - 2y = 40$ and therefore $x - y = 20$.

2014…

14. E Rearranging the equation $\frac{3x + y}{x - 3y} = -1$ gives $3x + y = -x + 3y$. So $4x = 2y$ and therefore $y = 2x$. Hence $\frac{x + 3y}{3x - y} = \frac{x + 3 \times 2x}{3x - 2x} = \frac{7x}{x} = 7$.

2015…

2. A Rearranging $6x = \frac{150}{x}$ gives $x^2 = \frac{150}{6}$, so $x^2 = 25$. This has two solutions, $x = 5$ and $x = -5$. Therefore the sum of the solutions is $5 + (-5) = 0$.

25. C Repeatedly using the rule that $f(xy) = f(x) + f(y)$ allows us to write $f(500)$ as $f(2 \times 2 \times 5 \times 5 \times 5) = f(2) + f(2) + f(5) + f(5) + f(5) = 2f(2) + 3f(5)$. We are given values for $f(40)$ and $f(10)$ and from them we need to calculate the values of $f(2)$ and $f(5)$. Now $f(40)$ can be written as $f(2) + f(2) + f(10)$ so $20 = 2f(2) + 14$ and therefore $f(2) = 3$. Similarly $f(10) = f(2) + f(5)$ so $14 = 3 + f(5)$ giving $f(5) = 11$. So $f(500) = 2f(2) + 3f(5) = 2 \times 3 + 3 \times 11 = 39$.

2016…

9. C Let $O = (0,0)$, $A = (1,0)$, $B = (1,1)$, $C = (0,1)$ be the vertices of the square. The equation $x^2 + y^2 = 1$ gives a circle passing through A and C. The equation $y = x + 1$ gives a straight line passing only through C. The equation $y = -x^2 + 1$ gives a parabola passing through A and C. The equation $y = x$ gives a straight line passing through O and B. The equation $y = \frac{1}{x}$ gives a rectangular hyperbola which has two branches and passes only through B. So, only $x^2 + y^2 = 1$, $y = x$ and $y = -x^2 + 1$ have graphs passing through exactly two of the vertices of the square.