UKMT 3D Shapes Questions

(Answers follow after all the questions)

2005…

18. A cube exactly fits inside a sphere and another sphere exactly fits inside this cube. What is the ratio of the volume of the smaller sphere to the volume of the larger sphere?

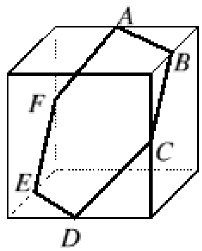
A $1 : 3\sqrt{3}$ B $1 : 4$ C $1 : 3$ D $2 : 3$ E $1 : 2$

2006…

7. The base of a pyramid has n edges. What is the difference between the number of edges the pyramid has and the number of faces the pyramid has?

A $n - 2$ B $n - 1$ C n D $n + 1$ E $n + 2$

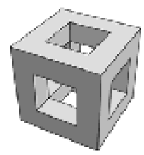
24. A solid red plastic cube, volume $1\ \text{cm}^3$, is painted white on its outside. The cube is cut by a plane passing through the mid-points of various edges, as shown.
What, in cm^2, is the *total* red area exposed by the cut?



A $\frac{3\sqrt{3}}{2}$ B 2 C $\frac{9\sqrt{2}}{5}$ D 3 E $\frac{3(\sqrt{3} + \sqrt{2})}{4}$

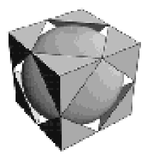
2007…

11. A $4 \times 4 \times 4$ cube has three $2 \times 2 \times 4$ holes drilled symmetrically all the way through, as shown.
What is the surface area of the resulting solid?



A 192 B 144 C 136 D 120 E 96

24. A paperweight is made from a glass cube of side 2 units by first shearing off the eight tetrahedral corners which touch at the midpoints of the edges of the cube. The remaining inner core of the cube is discarded and replaced by a sphere. The eight corner pieces are now stuck onto the sphere so that they have the same positions relative to each other as they did originally. What is the diameter of the sphere?



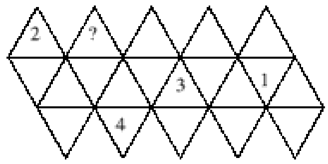
A $\sqrt{8} - 1$ B $\sqrt{8} + 1$ C $\frac{1}{3}(6 + \sqrt{3})$ D $\frac{4}{3}\sqrt{3}$ E $2\sqrt{3}$

2009…

17. A solid cube is divided into two pieces by a single rectangular cut. As a result, the total surface area increases by a fraction f of the surface area of the original cube. What is the greatest possible value of f?

A $\frac{1}{3}$ B $\frac{\sqrt{3}}{4}$ C $\frac{\sqrt{2}}{3}$ D $\frac{1}{2}$ E $\frac{1}{\sqrt{3}}$

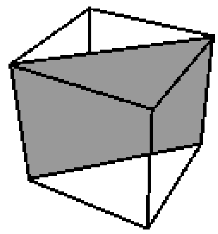
23. The net shown is folded into an icosahedron and the remaining faces are numbered such that at each vertex the numbers 1 to 5 all appear. What number must go on the face with a question mark?



A 1 B 2 C 3 D 4 E 5

2010…

18. A solid cube of side 2 cm is cut into two triangular prisms by a plane passing through four vertices, as shown. What is the total surface area of these two prisms?



A $8(3 + \sqrt{2})$ B $2(8 + \sqrt{2})$ C $8(3 + 2\sqrt{2})$ D $16(3 + \sqrt{2})$ E $8\sqrt{2}$

24. Three spheres of radius 1 are placed on a horizontal table and inside a vertical hollow cylinder of height 2 units which is just large enough to surround them. What fraction of the internal volume of the cylinder is occupied by the spheres?

A $\frac{2}{7 + 4\sqrt{3}}$ B $\frac{2}{2 + \sqrt{3}}$ C $\frac{1}{3}$ D $\frac{3}{2 + \sqrt{3}}$ E $\frac{6}{7 + 4\sqrt{3}}$

2011…

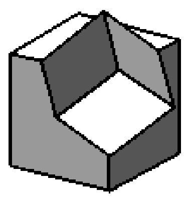
9. Sam has a large collection of $1 \times 1 \times 1$ cubes, each of which is either red or yellow. Sam makes a $3 \times 3 \times 3$ block from twenty-seven cubes, so that no cubes of the same colour meet face-to-face.

What is the difference between the largest number of red cubes that Sam can use and the smallest number?

A 0 B 1 C 2 D 3 E 4

25. A solid sculpture consists of a $4 \times 4 \times 4$ cube with a $3 \times 3 \times 3$ cube sticking out, as shown. Three vertices of the smaller cube lie on edges of the larger cube, the same distance along each.

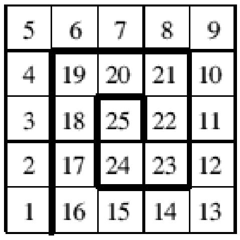
What is the total volume of the sculpture?



A 79 B 81 C 82 D 84 E 85

2012…

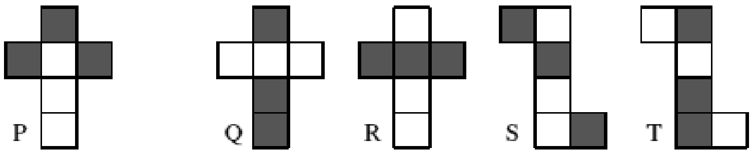
13. A cube is placed with one face on square 1 in the maze shown, so that it completely covers the square with no overlap. The upper face of the cube is covered in wet paint. The cube is then ‘rolled’ around the maze, rotating about an edge each time, until it reaches square 25. It leaves paint on all of the squares on which the painted face lands, but on no others. The cube is removed on reaching the square 25. What is the sum of the numbers on the squares which are now marked with paint?



A 78 B 80 C 82 D 169 E 625

2014…

13. Each of the five nets P, Q, R, S and T is made from six squares. Both sides of each square have the same colour. Net P is folded to form a cube.



How many of the nets Q, R, S and T can be folded to produce a cube that looks the same as that produced by P?

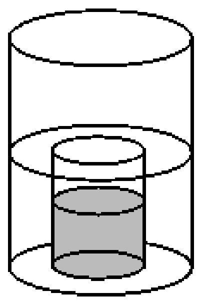
A 0 B 1 C 2 D 3 E 4

2015…

15. Two vases are cylindrical in shape. The larger vase has diameter 20 cm. The smaller vase has diameter 10 cm and height 16 cm. The larger vase is partially filled with water. Then the empty smaller vase, with the open end at the top, is slowly pushed down into the water, which flows over its rim. When the smaller vase is pushed right down, it is half full of water.

What was the original depth of the water in the larger vase?

A 10 cm B 12 cm C 14 cm D 16 cm E 18 cm



2016…

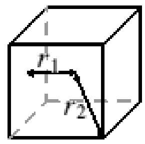
23. A cuboid has sides of lengths 22, 2 and 10. It is contained within a sphere of the smallest possible radius. What is the side-length of the largest cube that will fit inside the same sphere?

A 10 B 11 C 12 D 13 E 14

UKMT 3D Shapes Answers

2005…

18. **A** Let the radii of the two spheres be r_1 and r_2, as shown. Applying Pythagoras' Theorem: $r_2^2 = r_1^2 + r_1^2 + r_1^2$, so $r_2 = \sqrt{3}r_1$. The ratio of the volumes of the spheres $= r_1^3 : r_2^3 = 1 : (\sqrt{3})^3$, that is $1 : 3\sqrt{3}$.



2006…

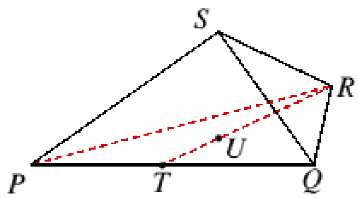
7. **B** The pyramid has $2n$ edges and $n + 1$ faces, so the required difference is $2n - (n + 1)$, that is $n - 1$.

24. **A** Let O be the centre of the cube. Consider triangle ABO: from Pythagoras' Theorem, $OA = AB = BO = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$ cm $= \frac{1}{\sqrt{2}}$ cm. So triangle OAB is equilateral. A similar argument may be applied to triangles OBC, OCD etc. The area of each of these equilateral triangles is $\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \sin 60°$ cm^2, that is $\frac{1}{8}\sqrt{3}$ cm^2. So the area of hexagon $ABCDEF$ is $6 \times \frac{\sqrt{3}}{8}$ cm^2. However, the total red area exposed by the cut is twice the area of this hexagon, that is $\frac{3\sqrt{3}}{2}$ cm^2.

2007…

11. **D** Each of the original faces of the cube now has area $4 \times 4 - 2 \times 2$, that is 12. In addition, the drilling of the holes has created 24 rectangles, each measuring 2×1. So the required area is $6 \times 12 + 24 \times 2 = 120$.

24. **D** The diameter of the sphere is $l - 2h$ where l is the length of a space diagonal of the cube and h is the perpendicular height of one of the tetrahedral corners when its base is an equilateral triangle.



The diagram shows such a tetrahedron: S is a corner of the cube; the base of the tetrahedron, which is considered to lie in a horizontal plane, is an equilateral triangle, PQR, of side $\sqrt{2}$ units; T is the midpoint of PQ. Also U is the centroid of triangle PQR, so $RU : UT = 2 : 1$. As U is vertically below S, the perpendicular height of the tetrahedron is SU.

As RTP is a right angle, $RT^2 = RP^2 - TP^2 = (\sqrt{2})^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{3}{2}$. Also, $RU = \frac{2}{3}RT$, so $RU^2 = \frac{4}{9}RT^2 = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$.

So $SU^2 = SR^2 - RU^2 = 1 - \frac{2}{3} = \frac{1}{3}$. Therefore $h = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$.

Now $l^2 = 2^2 + 2^2 + 2^2 = 12$, so $l = \sqrt{12} = 2\sqrt{3}$. Therefore the diameter of the sphere is $2\sqrt{3} - 2 \times \dfrac{\sqrt{3}}{3} = \dfrac{4\sqrt{3}}{3}$.

2009…

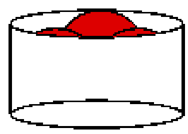
17. **C** The greatest possible value of f is achieved by a rectangular cut through an edge of a cube and the furthest edge from it. If we take x as the side of the cube, by Pythagoras' Theorem the extra surface area formed by the cut is $2\sqrt{2}x^2$. Hence $f = \frac{2\sqrt{2}x^2}{6x^2} = \frac{\sqrt{2}}{3}$.

23. **D** Label the rows of the triangles from left to right as follows: $a_1, \ldots, a_5$; $b_1, \ldots, b_{10}$ and $c_1, \ldots, c_5$.
Now 1 cannot be at a_4, a_5, b_7, b_8 or c_4 hence 1 must be at c_3.
Hence b_4 and b_5 are 2 and 5 in either order. Hence a_3 is 1 or 4.
But 1 cannot be at a_4 or b_7 hence 1 must be at a_3.
4 cannot be at b_3 thus 4 is at a_2.
Hence the number on the face with the question mark must be 4.

2010…

18. **A** Let x be the length of the shaded rectangle.
By Pythagoras' Theorem, $x^2 = 2^2 + 2^2$, hence $x = 2\sqrt{2}$.
The total surface area of the two prisms equals the surface area of the solid cube plus twice the surface area of that shaded rectangle, that is $6 \times 2 \times 2 + 2 \times 2 \times 2\sqrt{2} = 24 + 8\sqrt{2} = 8(3 + \sqrt{2})$.

24. **E** The volume of the three spheres is $3 \times \frac{4}{3}\pi \times 1^3 = 4\pi$.



Let r be the radius of the cross-sectional area of the cylinder.
Hence the volume of the cylinder is $2\pi r^2$.
Thus the required fraction is $\frac{2}{r^2}$.
The straight lines joining the centres of the three spheres form an equilateral triangle of side length 2.
Let x be the distance from the centre of a sphere to the midpoint of the triangle. Using the Sine Rule, $\frac{2}{\sin 120°} = \frac{x}{\sin 30°}$ hence $x = \frac{2}{\sqrt{3}}$.
As the sphere has radius 1, $r = x + 1$ and $r = 1 + \frac{2}{\sqrt{3}}$.
Thus $r^2 = \frac{1}{3}(2 + \sqrt{3})^2 = \frac{1}{3}(7 + 4\sqrt{3})$. Hence the required fraction is $\frac{6}{7 + 4\sqrt{3}}$.

2011…

9. **B** Let the centre cube in the $3 \times 3 \times 3$ block be red. As no cubes of the same colour meet face-to-face then the 6 centre cubes on the outer faces must be yellow. All six outer faces are as shown alongside.

Y	R	Y
R	Y	R
Y	R	Y

Thus 14 faces are yellow and 13 faces are red. If the centre cube is yellow then the situation is reversed. Hence the difference between the largest number of red cubes that Sam can use and the smallest number is 1.

25. C Three vertices of the smaller cube lie on edges of the larger cube, the same distance along each. Let this distance be x and let the distance between any two of these vertices be y. Hence, by Pythagoras' Theorem, $y^2 = x^2 + x^2$ and, as the side length of the smaller cube is 3, $y^2 = 3^2 + 3^2$. Thus $x = 3$ and $y = 3\sqrt{2}$.
The intersection of the cubes forms two congruent tetrahedra of base area equal to $\frac{1}{2}y^2 \sin 60° = \frac{1}{2}(3\sqrt{2})^2 \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$. Let h be the perpendicular height of the tetrahedra. Hence, using Pythagoras' Theorem twice gives $9 = h^2 + 6$, thus $h = \sqrt{3}$.
Thus the total volume of the sculpture is $4^3 + 3^3 - 2 \times \frac{1}{3} \times \frac{9\sqrt{3}}{2} \times \sqrt{3} = 91 - 9 = 82$.

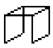
2012…

13. B The table below shows the position of the face marked with paint when the base of the cube is on the 25 squares. Code: T - top, B - base; F - front; H - hidden (rear); L - left; R - right.

1	2	**3**	4	5	6	**7**	8	9	10	**11**	12	13	14	**15**	16	17	18	19	**20**	21	22	23	**24**	25
T	H	**B**	F	T	R	**B**	L	T	F	**B**	H	T	L	**B**	R	R	R	R	**B**	L	L	L	**B**	F

So the required sum is 3 + 7 + 11 + 15 + 20 + 24 = 80.

2014…

13. E Each of P, Q, R, S and T when folded to form a cube consists of a  shape of three black faces and an interlocking  shape of three white faces, so they are all nets of the same cube.

2015…

15. C Let the original water level in the larger vase be h cm. The volume of water at the start is then $\pi \times 10^2 \times h$ cm^3. The volume of water completely within the vase is constant, but when the smaller vase is pushed down, some of the water moves into it. In the end the depth of the water in the larger vase is the same as the height of the smaller vase itself, which is 16 cm. We are given that the final depth of water in the smaller vase is 8 cm. So the total volume of water is then $\pi \times 10^2 \times 16$ cm^3 less the gap in the top half of the smaller vase. So $\pi \times 10^2 \times h = \pi \times 10^2 \times 16 - \pi \times 5^2 \times 8$, giving $100\pi h = 1600\pi - 200\pi$ and therefore $h = 14$.

2016…

23. E For the cuboid to be contained within a sphere of smallest possible radius, all eight vertices of the cuboid must lie on the sphere. The radius r of the smallest sphere is then half of the length of the body diagonal of the cuboid, so $r = \sqrt{1^2 + 5^2 + 11^2} = \sqrt{147}$. If the largest cube which will fit inside this sphere has side-length $2x$, then $r = \sqrt{x^2 + x^2 + x^2}$. Thus $3x^2 = 147$, so $x^2 = 49$ and so $x = 7$. The side-length of the largest cube is 14.