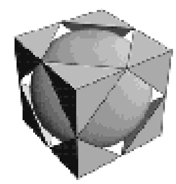
24. A paperweight is made from a glass cube of side 2 units by first shearing off the eight tetrahedral corners which touch at the midpoints of the edges of the cube. The remaining inner core of the cube is discarded and replaced by a sphere. The eight corner pieces are now stuck onto the sphere so that they have the same positions relative to each other as they did originally. What is the diameter of the sphere?



A $\sqrt{8} - 1$ B $\sqrt{8} + 1$ C $\frac{1}{3}(6 + \sqrt{3})$ D $\frac{4}{3}\sqrt{3}$ E $2\sqrt{3}$

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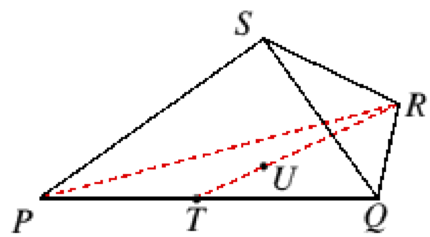
24. D The diameter of the sphere is $l - 2h$ where l is the length of a space diagonal of the cube and h is the perpendicular height of one of the tetrahedral corners when its base is an equilateral triangle.



The diagram shows such a tetrahedron: S is a corner of the cube; the base of the tetrahedron, which is considered to lie in a horizontal plane, is an equilateral triangle, PQR, of side $\sqrt{2}$ units; T is the midpoint of PQ. Also U is the centroid of triangle PQR, so $RU : UT = 2 : 1$. As U is vertically below S, the perpendicular height of the tetrahedron is SU.

As RTP is a right angle, $RT^2 = RP^2 - TP^2 = (\sqrt{2})^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{3}{2}$. Also, $RU = \frac{2}{3}RT$, so $RU^2 = \frac{4}{9}RT^2 = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$.

So $SU^2 = SR^2 - RU^2 = 1 - \frac{2}{3} = \frac{1}{3}$. Therefore $h = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$.

Now $l^2 = 2^2 + 2^2 + 2^2 = 12$, so $l = \sqrt{12} = 2\sqrt{3}$. Therefore the diameter of the sphere is $2\sqrt{3} - 2 \times \dfrac{\sqrt{3}}{3} = \dfrac{4\sqrt{3}}{3}$.