Core 2 Trigonometry Questions
(From the Oxford MAT Tests)

For answers, see the MAT website

Specimen A, Question 1f:

F. How many values of x satisfy the equation

$$2\cos^2 x + 5\sin x = 4$$

in the range $0 \leqslant x < 2\pi$?

(a) 2 (b) 4 (c) 6 (d) 8

Specimen A, Question 2:

(i) Show, with working, that

$$x^3 - (1 + \cos\theta + \sin\theta)\,x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)\,x - \sin\theta\cos\theta, \qquad (1)$$

equals

$$(x-1)\left(x^2 - (\cos\theta + \sin\theta)\,x + \cos\theta\sin\theta\right)$$

Deduce that the cubic in (1) has roots

$$1, \qquad \cos\theta, \qquad \sin\theta.$$

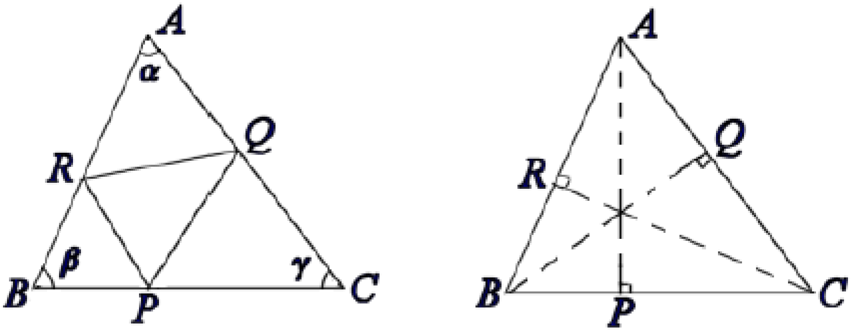
(ii) Give the roots when $\theta = \frac{\pi}{3}$.

(iii) Find all values of θ in the range $0 \leqslant \theta < 2\pi$ such that two of the three roots are equal.

(iv) What is the greatest possible difference between two of the roots, and for what values of θ in the range $0 \leqslant \theta < 2\pi$ does this greatest difference occur?

Show that for each such θ the cubic (1) is the same.

Specimen A, Question 4:



A triangle ABC has sides BC, CA and AB of sides a, b and c respectively, and angles at A, B and C are α, β and γ where $0 \leqslant \alpha, \beta, \gamma \leqslant \frac{1}{2}\pi$.

(i) Show that the area of ABC equals $\frac{1}{2}bc\sin\alpha$.

Deduce the sine rule

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

(ii) The points P, Q and R are respectively the feet of the perpenicular from A to BC, B to CA, and C to AB as shown.

Prove that

$$\text{Area of } PQR = \left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma\right) \times (\text{Area of } ABC).$$

(iii) For what triangles ABC, with angles α, β, γ as above, does the equation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

hold?

Specimen B, Question 1c:

C. Which of the following numbers is largest in value? (All angles are given in radians.)

(a) $\tan\left(\frac{5\pi}{4}\right)$ (b) $\sin^2\left(\frac{5\pi}{4}\right)$ (c) $\log_{10}\left(\frac{5\pi}{4}\right)$ (d) $\log_2\left(\frac{5\pi}{4}\right)$

Specimen B, Question 1e:

E. In the range $0 \leqslant x < 2\pi$ the equation

$$\cos(\sin x) = \frac{1}{2}$$

has

(a) no solutions;
(b) one solution;
(c) two solutions;
(d) three solutions.

2007, Question 1b:

B. The greatest value which the function

$$f(x) = \left(3\sin^2(10x + 11) - 7\right)^2$$

takes, as x varies over all real values, equals

(a) -9, (b) 16, (c) 49, (d) 100.

2007, Question 1c:

C. The number of solutions x to the equation

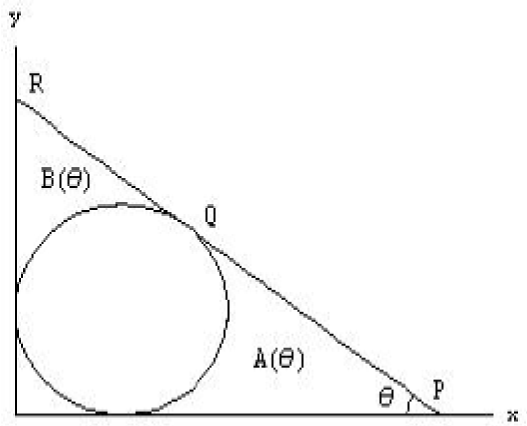
$$7\sin x + 2\cos^2 x = 5,$$

in the range $0 \leqslant x < 2\pi$, is

(a) 1, (b) 2, (c) 3, (d) 4.

2007, Question 4:

In the diagram below is sketched the circle with centre $(1,1)$ and radius 1 and a line L. The line L is tangential to the circle at Q; further L meets the y-axis at R and the x-axis at P in such a way that the angle OPQ equals θ where $0 < \theta < \pi/2$.



(i) Show that the co-ordinates of Q are

$$(1 + \sin\theta, 1 + \cos\theta),$$

and that the gradient of PQR is $-\tan\theta$.

Write down the equation of the line PQR and so find the co-ordinates of P.

(ii) The region bounded by the circle, the x-axis and PQ has area $A(\theta)$; the region bounded by the circle, the y-axis and QR has area $B(\theta)$. (See diagram.)

Explain why

$$A(\theta) = B(\pi/2 - \theta)$$

for any θ.

Calculate $A(\pi/4)$.

(iii) Show that

$$A\left(\frac{\pi}{3}\right) = \sqrt{3} - \frac{\pi}{3}.$$

2008, Question 1c:

C. The simultaneous equations in x, y,

$$\begin{aligned} (\cos\theta)\,x - (\sin\theta)\,y &= 2 \\ (\sin\theta)\,x + (\cos\theta)\,y &= 1 \end{aligned}$$

are solvable

(a) for all values of θ in the range $0 \leqslant \theta < 2\pi$;
(b) except for one value of θ in the range $0 \leqslant \theta < 2\pi$;
(c) except for two values of θ in the range $0 \leqslant \theta < 2\pi$;
(d) except for three values of θ in the range $0 \leqslant \theta < 2\pi$.

2008, Question 1j:

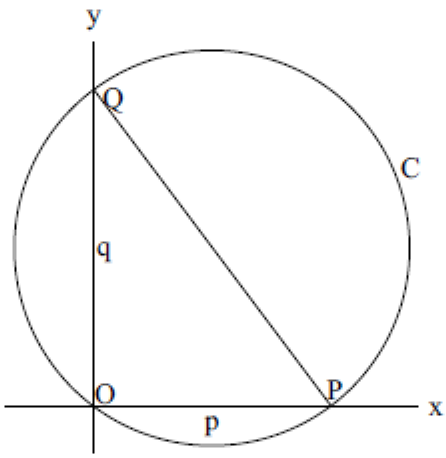
J. In the range $0 \leqslant x < 2\pi$ the equation

$$(3 + \cos x)^2 = 4 - 2\sin^8 x$$

has

(a) 0 solutions, (b) 1 solution, (c) 2 solutions, (d) 3 solutions.

2008, Question 4:



Let p and q be positive real numbers. Let P denote the point $(p, 0)$ and Q denote the point $(0, q)$.

(i) Show that the equation of the circle C which passes through P, Q and the origin O is

$$x^2 - px + y^2 - qy = 0.$$

Find the centre and area of C.

(ii) Show that

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} \geqslant \pi.$$

(iii) Find the angles OPQ and OQP if

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = 2\pi.$$

2009, Question 1e:

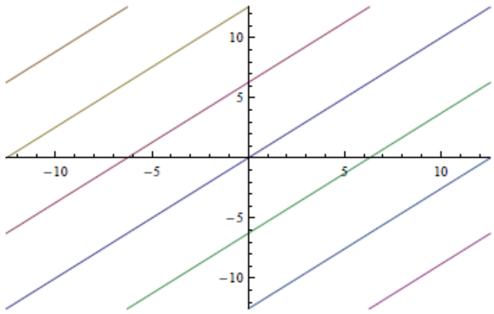
E. In the range $0 \leqslant x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

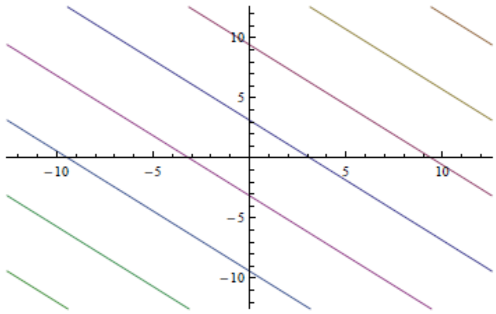
(a) has 0 solutions;
(b) has 1 solution;
(c) has 2 solutions;
(d) holds for all values of x.

2009, Question 1g:

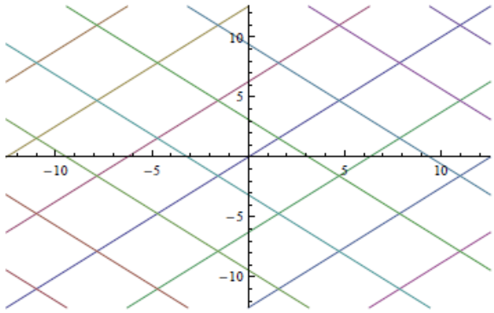
G. The graph of all those points (x, y) in the xy-plane which satisfy the equation $\sin y = \sin x$ is drawn in



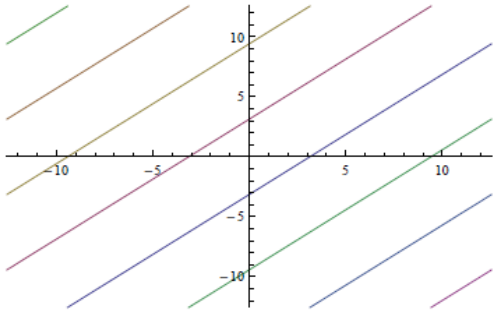
(a)



(b)



(c)



(d)

2010, Question 1c:

C. In the range $0 \leqslant x < 2\pi$, the equation

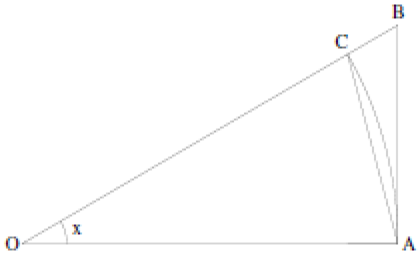
$$\sin^2 x + 3\sin x \cos x + 2\cos^2 x = 0$$

has

(a) 1 solution, (b) 2 solutions, (c) 3 solutions, (d) 4 solutions.

2010, Question 3:

[In this question, you may assume that the derivative of $\sin x$ is $\cos x$.]



(i) In the diagram above OA and OC are of length 1 and subtend an angle x at O. The angle BAO is a right angle and the circular arc from A to C, centred at O, is also drawn.

By consideration of various areas in the above diagram, show, for $0 < x < \pi/2$, that

$$x \cos x < \sin x < x.$$

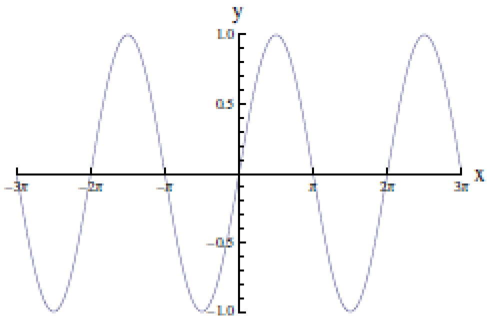
(ii) Sketch, on the axes provided on the opposite page, the graph of

$$y = \frac{\sin x}{x}, \qquad 0 < x < 4\pi.$$

Justify your value that y takes as x becomes small.

[You do not need to determine the coordinates of the turning points.]

(iii) Drawn below is a graph of $y = \sin x$. Sketch on the same axes the line $y = cx$ where $c > 0$ is such that the equation $\sin x = cx$ has *exactly 5 solutions*.



(iv) Draw the line $y = c$ on the axes on the opposite page.

(v) If X is the largest of the five solutions of the equation $\sin x = cx$, explain why $\tan X = X$.

2010, Question 4:

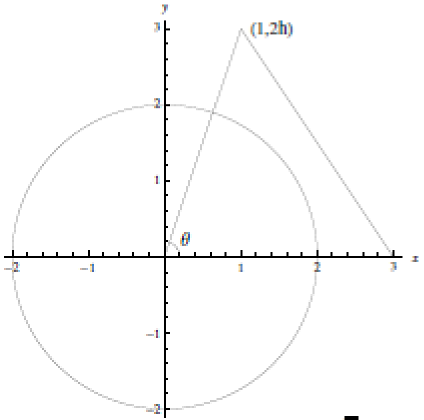


Diagram when $h > 2/\sqrt{5}$

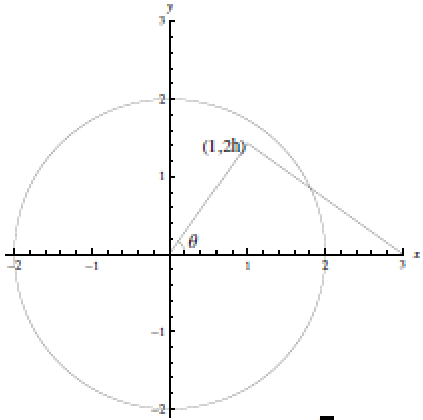


Diagram when $h < \sqrt{3}/2$

The three corners of a triangle T are $(0, 0)$, $(3, 0)$, $(1, 2h)$ where $h > 0$. The circle C has equation $x^2 + y^2 = 4$. The angle of the triangle at the origin is denoted as θ. The circle and triangle are drawn in the diagrams above for different values of h.

(i) Express $\tan\theta$ in terms of h.

(ii) Show that the point $(1, 2h)$ lies inside C when $h < \sqrt{3}/2$.

(iii) Find the equation of the line connecting $(3, 0)$ and $(1, 2h)$. Show that this line is tangential to the circle C when $h = 2/\sqrt{5}$.

(iv) Suppose now that $h > 2/\sqrt{5}$. Find the area of the region inside both C and T in terms of θ.

(v) Now let $h = 6/7$. Show that the point $(8/5, 6/5)$ lies on both the line (from part (iii)) and the circle C.

Hence show that the area of the region inside both C and T equals

$$\frac{27}{35} + 2\alpha$$

where α is an angle whose tangent, $\tan\alpha$, you should determine.

[You may use the fact that the area of a triangle with corners $(0, 0)$, (a, b), (c, d) equals $\frac{1}{2}|ad - bc|$.]

2011, Question 1d:

D. The fraction of the interval $0 \leqslant x \leqslant 2\pi$, for which one (or both) of the inequalities

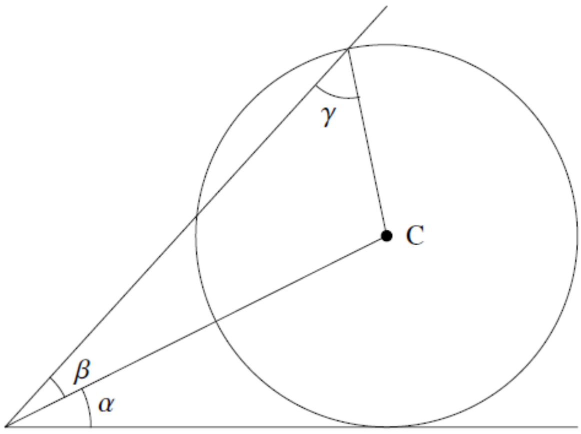
$$\sin x \geqslant \frac{1}{2}, \qquad \sin 2x \geqslant \frac{1}{2}$$

is true, equals

(a) $\frac{1}{3}$, (b) $\frac{13}{24}$, (c) $\frac{7}{12}$, (d) $\frac{5}{8}$.

2011, Question 1e:

E. The circle in the diagram has centre C. Three angles α, β, γ are also indicated.



The angles α, β, γ are related by the equation:

(a) $\cos\alpha = \sin(\beta+\gamma)$;
(b) $\sin\beta = \sin\alpha\sin\gamma$;
(c) $\sin\beta(1-\cos\alpha) = \sin\gamma$;
(d) $\sin(\alpha+\beta) = \cos\gamma\sin\alpha$.

2011, Question 1f:

F. Given θ in the range $0 \leqslant \theta < \pi$, the equation

$$x^2 + y^2 + 4x\cos\theta + 8y\sin\theta + 10 = 0$$

represents a circle for

(a) $0 < \theta < \frac{\pi}{3}$, (b) $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, (c) $0 < \theta < \frac{\pi}{2}$, (d) all values of θ.

2011, Question 1i:

I. In the range $0 \leqslant x < 2\pi$ the equation

$$\sin^8 x + \cos^6 x = 1$$

has

(a) 3 solutions , (b) 4 solutions, (c) 6 solutions, (d) 8 solutions.

2012, Question 1f:

F. Let

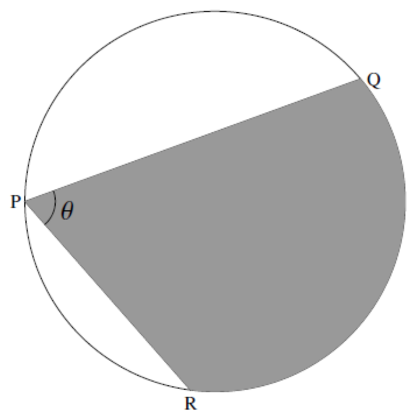
$$T = \left(\int_{-\pi/2}^{\pi/2} \cos x \, \mathrm{d}x\right) \times \left(\int_{\pi}^{2\pi} \sin x \, \mathrm{d}x\right) \times \left(\int_{0}^{\pi/8} \frac{\mathrm{d}x}{\cos 3x}\right).$$

Which of the following is true?

(a) $T = 0$; (b) $T < 0$; (c) $T > 0$; (d) T is not defined.

2012, Question 1j:

J. If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below,

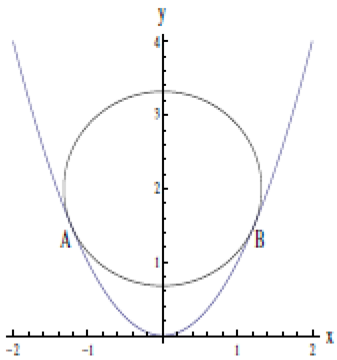


then the largest possible area of the shaded region RPQ is

(a) $\theta\left(1 + \cos\left(\frac{\theta}{2}\right)\right)$; (b) $\theta + \sin\theta$; (c) $\frac{\pi}{2}(1 - \cos\theta)$; (d) θ.

2012, Question 4:

The diagram below shows the parabola $y = x^2$ and a circle with centre $(0, 2)$ just 'resting' on the parabola. By 'resting' we mean that the circle and parabola are tangential to each other at the points A and B.



(i) Let (x, y) be a point on the parabola such that $x \neq 0$. Show that the gradient of the line joining this point to the centre of the circle is given by

$$\frac{x^2 - 2}{x}.$$

(ii) With the help of the result from part (i), or otherwise, show that the coordinates of B are given by

$$\left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right).$$

(iii) Show that the area of the sector of the circle enclosed by the radius to A, the minor arc AB and the radius to B is equal to

$$\frac{7}{4}\cos^{-1}\left(\frac{1}{\sqrt{7}}\right).$$

(iv) Suppose now that a circle with centre $(0, a)$ is resting on the parabola, where $a > 0$. Find the range of values of a for which the circle and parabola touch at two distinct points.

(v) Let r be the radius of a circle with centre $(0, a)$ that is resting on the parabola. Express a as a function of r, distinguishing between the cases in which the circle is, and is not, in contact with the vertex of the parabola.

2013, Question 1b:

B. The graph of $y = \sin x$ is reflected first in the line $x = \pi$ and then in the line $y = 2$. The resulting graph has equation

(a) $y = \cos x$; (b) $y = 2 + \sin x$; (c) $y = 4 + \sin x$; (d) $y = 2 - \cos x$.

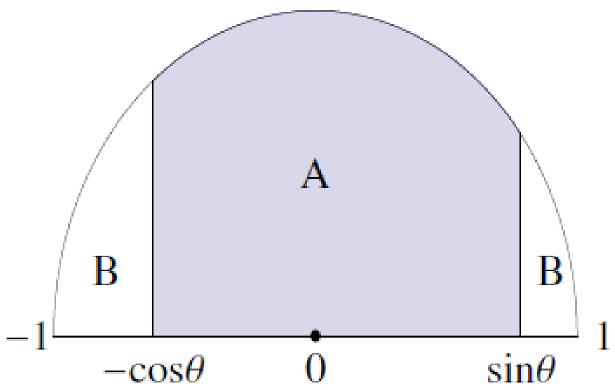
2013, Question 4:

(i) Let $a > 0$. On the axes opposite, sketch the graph of

$$y = \frac{a+x}{a-x} \qquad \text{for} \qquad -a < x < a.$$

(ii) Let $0 < \theta < \pi/2$. In the diagram below is the half-disc given by $x^2 + y^2 \leqslant 1$ and $y \geqslant 0$. The shaded region A consists of those points with $-\cos\theta \leqslant x \leqslant \sin\theta$. The region B is the remainder of the half-disc.

Find the area of A.



(iii) Assuming *only* that $\sin^2\theta + \cos^2\theta = 1$, show that $\sin\theta\cos\theta \leqslant 1/2$.

(iv) What is the largest that the ratio

$$\frac{\text{area of } A}{\text{area of } B}$$

can be, as θ varies?

2014, Question 1e:

E. As x varies over the real numbers, the largest value taken by the function

$$\left(4\sin^2 x + 4\cos x + 1\right)^2$$

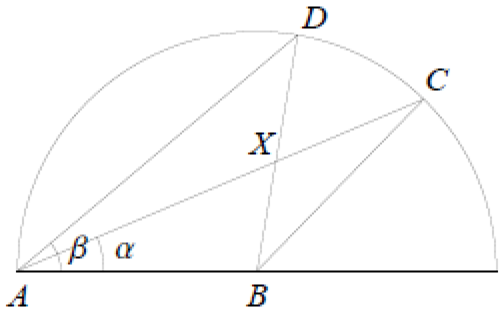
equals

(a) $17+12\sqrt{2}$, (b) 36, (c) $48\sqrt{2}$, (d) $64-12\sqrt{3}$, (e) 81.

2014, Question 4:

In the diagram below is sketched a semicircle with centre B and radius 1. Three points A, C, D lie on the semicircle as shown with α denoting angle CAB and β denoting angle DAB. The triangles ABC and ABD intersect in a triangle ABX.

Throughout the question we shall consider the value of α fixed. Assume for now that $0 < \alpha \leqslant \beta \leqslant \pi/2$.



(i) Show that the area of the triangle ABC equals

$$\frac{1}{2}\sin(2\alpha).$$

(ii) Let

$$F = \frac{\text{area of triangle } ABX}{\text{area of triangle } ABC}.$$

Without calculation, explain why, for every k in the range $0 \leqslant k \leqslant 1$, there is a unique value of β such that $F = k$.

(iii) Find the value of β such that $F = 1/2$.

(iv) Show that

$$F = \frac{\sin(2\beta)\sin\alpha}{\sin(2\beta - \alpha)\sin(2\alpha)}.$$

(v) Suppose now that $0 < \beta < \alpha \leqslant \pi/2$. Write down, without further calculation, an expression for the area of ABX and hence a formula for F.