Core 2 Sequences Questions
(From the Oxford MAT Tests)

For answers, see [the MAT website]

2007, Question 1j:

J. The inequality

$$(n+1)+(n^4+2)+(n^9+3)+(n^{16}+4)+\cdots+(n^{10000}+100)>k$$

is true for all $n \geqslant 1$. It follows that

(a) $k < 1300$,
(b) $k^2 < 101$,
(c) $k \geqslant 101^{10000}$,
(d) $k < 5150$.

2008, Question 1i:

I. The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n.$$

For example, $S(723) = 7+2+3 = 12$. The sum

$$S(1)+S(2)+S(3)+\cdots+S(99)$$

equals

(a) 746, (b) 862, (c) 900, (d) 924.

2008, Question 2:

(i) Find a pair of positive integers, x_1 and y_1, that solve the equation

$$(x_1)^2 - 2(y_1)^2 = 1.$$

(ii) Given integers a, b, we define two sequences $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ by setting

$$x_{n+1} = 3x_n + 4y_n, \qquad y_{n+1} = ax_n + by_n, \qquad \text{for } n \geqslant 1.$$

Find *positive* values for a, b such that

$$(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2.$$

(iii) Find a pair of integers X, Y which satisfy $X^2 - 2Y^2 = 1$ such that $X > Y > 50$.

(iv) (Using the values of a and b found in part (ii)) what is the approximate value of x_n/y_n as n increases?

2009, Question 1e:

E. In the range $0 \leqslant x < 2\pi$, the equation

$$2^{\sin^2 x} + 2^{\cos^2 x} = 2$$

(a) has 0 solutions;
(b) has 1 solution;
(c) has 2 solutions;
(d) holds for all values of x.

2009, Question 2:

A list of real numbers $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 1$, $x_2 = 3$ and then for $n \geqslant 3$ by

$$x_n = 2x_{n-1} - x_{n-2} + 1.$$

So, for example,

$$x_3 = 2x_2 - x_1 + 1 = 2 \times 3 - 1 + 1 = 6.$$

(i) Find the values of x_4 and x_5.

(ii) Find values of real constants A, B, C such that for $n = 1, 2, 3$,

$$x_n = A + Bn + Cn^2. \qquad (*)$$

(iii) Assuming that equation $(*)$ holds true for all $n \geqslant 1$, find the smallest n such that $x_n \geqslant 800$.

(iv) A second list of real numbers $y_1, y_2, y_3, \ldots$ is defined by $y_1 = 1$ and

$$y_n = y_{n-1} + 2n$$

Find, explaining your reasoning, a formula for y_n which holds for $n \geqslant 2$.

What is the approximate value of x_n/y_n for large values of n?

2010, Question 1b:

B. The sum of the first $2n$ terms of

$$1, 1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \frac{1}{16}, \ldots$$

is

(a) $2^n + 1 - 2^{1-n}$, (b) $2^n + 2^{-n}$, (c) $2^{2n} - 2^{3-2n}$, (d) $\dfrac{2^n - 2^{-n}}{3}$.

2011, Question 1c:

C. The sequence x_n is given by the formula

$$x_n = n^3 - 9n^2 + 631.$$

The largest value of n for which $x_n > x_{n+1}$ is

(a) 5, (b) 7, (c) 11, (d) 17.
