Problem Solving Questions
(From the Oxford MAT Tests)

For answers, see [the MAT website]

Specimen A, Question 5:

Songs of the Martian classical period had just two notes (let us call them x and y) and were constructed according to rigorous rules:

I. the sequence consisting of no notes was deemed to be a song (perhaps the most pleasant);

II. a sequence starting with x, followed by two repetitions of an existing song and ending with y was also a song;

III. the sequence of notes obtained by interchanging xs and ys in a song was also a song.

All songs were constructed using those rules.

(i) Write down four songs of length six (that is, songs with exactly six notes).

(ii) Show that if there are k songs of length m then there are $2k$ songs of length $2m + 2$. Deduce that for each natural number there are 2^n songs of length $2^{n+1} - 2$.

Songs of the Martian later period were constructed using also the rule:

IV. if a song ended in y then the sequence of notes obtained by omitting that y was also a song.

(iii) What lengths do songs of the later period have? That is, for which natural numbers n is there a song with exactly n notes? Justify your answer.

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Specimen B, Question 5:

An $n \times n$ square array contains 0s and 1s. Such a square is given below with $n = 3$.

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix}$$

Two types of operation C and R may be performed on such an array.

- The first operation C takes the first and second columns (on the left) and replaces them with a single column by comparing the two elements in each row as follows; if the two elements are the same the C replaces them with a 1, and if they differ C replaces them with a 0.
- The second operation R takes the first and second rows (from the top) and replaces them with a single row by comparing the two elements in each column as follows; if the two elements are the same the R replaces them with a 1, and if they differ R replaces them with a 0.

By way of example, the effects of performing R then C on the square above are given below.

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix} \xrightarrow{R} \begin{matrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{matrix} \xrightarrow{C} \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$$

(a) If R then C are performed on a 2×2 array then only a single number (0 or 1) remains.

(i) Write down in the grids on the next page the eight 2×2 arrays which, when R then C are performed, produce a 1.

(ii) By grouping your answers accordingly, show that if $\begin{matrix} a & b \\ c & d \end{matrix}$ is amongst your answers to part (i) then so is $\begin{matrix} a & c \\ b & d \end{matrix}$.

Explain why this means that doing R then C on a 2×2 array produces the same answer as doing C first then R.

(b) Consider now a $n \times n$ square array containing 0s and 1s, and the effects of performing R then C or C then R on the square.

(i) Explain why the effect on the right $n - 2$ columns is the same whether the order is R then C or C then R. [This then also applies to the bottom $n - 2$ rows.]

(ii) Deduce that performing R then C on an $n \times n$ square produces the same result as performing C then R.

Problem Solving Questions
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For answers, see (the MAT website)

2007, Question 5:

Let $f(n)$ be a function defined, for any integer $n \geqslant 0$, as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ (f(n/2))^2 & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 2f(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(i) What is the value of $f(5)$?

The *recursion depth* of $f(n)$ is defined to be the number of other integers m such that the value of $f(m)$ is calculated whilst computing the value of $f(n)$. For example, the recursion depth of $f(4)$ is 3, because the values of $f(2)$, $f(1)$, and $f(0)$ need to be calculated on the way to computing the value of $f(4)$.

(ii) What is the recursion depth of $f(5)$?

Now let $g(n)$ be a function, defined for all integers $n \geqslant 0$, as follows:

$$g(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 + g(n/2) & \text{if } n > 0 \text{ and } n \text{ is even}, \\ 1 + g(n-1) & \text{if } n > 0 \text{ and } n \text{ is odd}. \end{cases}$$

(iii) What is $g(5)$?

(iv) What is $g(2^k)$, where $k \geqslant 0$ is an integer? Briefly explain your answer.

(v) What is $g(2^l + 2^k)$ where $l > k \geqslant 0$ are integers? Briefly explain your answer.

(vi) Explain briefly why the value of $g(n)$ is equal to the recursion depth of $f(n)$.

Problem Solving Questions
(From the Oxford MAT Tests)

For answers, see (the MAT website)

2008, Question 5:

The Millennium school has 1000 students and 1000 student lockers. The lockers are in a line in a long corridor and are numbered from 1 to 1000.

Initially all the lockers are closed (but unlocked).

The first student walks along the corridor and opens every locker.

The second student then walks along the corridor and closes every second locker, i.e. closes lockers 2, 4, 6, etc. At that point there are 500 lockers that are open and 500 that are closed.

The third student then walks along the corridor, changing the state of every third locker. Thus s/he closes locker 3 (which had been left open by the first student), opens locker 6 (closed by the second student), closes locker 9, etc.

All the remaining students now walk by in order, with the kth student changing the state of every kth locker, and this continues until all 1000 students have walked along the corridor.

(i) How many lockers are closed immediately after the third student has walked along the corridor? Explain your reasoning.

(ii) How many lockers are closed immediately after the fourth student has walked along the corridor? Explain your reasoning.

(iii) At the end (after all 1000 students have passed), what is the state of locker 100? Explain your reasoning.

(iv) After the *hundredth* student has walked along the corridor, what is the state of locker 1000? Explain your reasoning.

Problem Solving Questions
(From the Oxford MAT Tests)

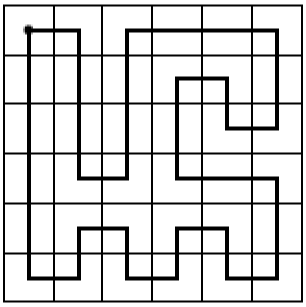
For answers, see (the MAT website)

2009, Question 5:

Given an $n \times n$ grid of squares, where $n > 1$, a *tour* is a path drawn within the grid such that:

- along its way the path moves, horizontally or vertically, from the centre of one square to the centre of an adjacent square;
- the path starts and finishes in the same square;
- the path visits the centre of every other square just once.

For example, below is a tour drawn in a 6×6 grid of squares which starts and finishes in the top-left square.



For parts (i)-(iv) it is assumed that n is *even*.

(i) With the aid of a diagram, show how a tour, which starts and finishes in the top-left square, can be drawn in any $n \times n$ grid.

(ii) Is a tour still possible if the start/finish point is changed to the centre of a different square? Justify your answer.

Suppose now that a robot is programmed to move along a tour of an $n \times n$ grid. The robot understands two commands:

- command R which turns the robot clockwise through a right angle;
- command F which moves the robot forward to the centre of the next square.

The robot has a program, a list of commands, which it performs in the given order to complete a tour; say that, in total, command R appears r times in the program and command F appears f times.

(iii) Initially the robot is in the top-left square pointing to the right. Assuming the first command is an F, what is the value of f? Explain also why $r + 1$ is a multiple of 4.

(iv) Must the results of part (iii) still hold if the robot starts and finishes at the centre of a different square? Explain your reasoning.

(v) Show that a tour of an $n \times n$ grid is not possible when n is odd.

Problem Solving Questions
(From the Oxford MAT Tests)

For answers, see (the MAT website)

2010, Question 5:

This question concerns calendar dates of the form

$$d_1d_2/m_1m_2/y_1y_2y_3y_4$$

in the order day/month/year.

The question specifically concerns those dates which contain no repetitions of a digit. For example, the date 23/05/1967 is one such date but 07/12/1974 is not such a date as both $1 = m_1 = y_1$ and $7 = d_2 = y_3$ are repeated digits.

We will use the Gregorian Calendar throughout (this is the calendar system that is standard throughout most of the world; see below.)

(i) Show that there is no date with no repetition of digits in the years from 2000 to 2099.

(ii) What was the last date before today with no repetition of digits? Explain your answer.

(iii) When will the next such date be? Explain your answer.

(iv) How many such dates were there in years from 1900 to 1999? Explain your answer.

[The Gregorian Calendar uses 12 months, which have, respectively, 31, 28 or 29, 31, 30, 31, 30, 31, 31, 30, 31, 30 and 31 days. The second month (February) has 28 days in years that are not divisible by 4, or that are divisible by 100 but not 400 (such as 1900); it has 29 days in the other years (leap years).]

Problem Solving Questions
(From the Oxford MAT Tests)

For answers, see (the MAT website)

2011, Question 5:

An $n \times n$ *grid* consists of squares arranged in n rows and n columns; for example, a chessboard is an 8×8 grid. Let us call a *semi-grid* of size n the lower left part of an $n \times n$ grid – that is, the squares located on or below the grid's diagonal. For example, Figure C shows an example of a semi-grid of size 4.

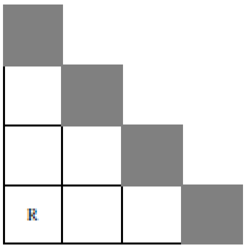


Figure C

Let us suppose that a robot is located in the lower-left corner of the grid. The robot can move only up or right, and its goal is to reach one of the *goal squares*, which are all located on the semi-grid's diagonal. In the example shown in Figure C, the robot is initially located in the square denoted with R, and the goal squares are shown in grey. Let us call a *solution* a sequence of the robot's moves that leads the robot from the initial location to some goal square.

(i) Write down all 8 solutions for a robot on a semi-grid of size 4.

(ii) Devise a concise way of representing the possible journeys of the robot in a semi-grid of size n. In your notation, which of the journeys are solutions?

(iii) Write down a formula for the number of possible solutions in a semi-grid of size n. Explain why your formula is correct.

Now let us change the problem slightly and redefine a goal square as any square that can be described as follows:

- the lower-left square is not a goal square;
- each square that is located immediately above or immediately to the right of a non-goal square is a goal square; and
- each square that is located immediately above or immediately to the right of a goal square is a non-goal square.

Furthermore, let us assume that, upon reaching a goal square, the robot may decide to stop or to continue moving (provided that there are more allowed moves).

(iv) With these modifications in place, write down all the solutions in a semi-grid of size 4, and all the solutions in a semi-grid of size 5.

(v) How many solutions are there now in a semi-grid of size n, where n is a positive integer? You may wish to consider separately the cases where n is even or odd.

Problem Solving Questions
(From the Oxford MAT Tests)

For answers, see (the MAT website)

2012, Question 5:

A particular robot has three commands:

F: Move forward a unit distance;
L: Turn left 90°;
R: Turn right 90°.

A *program* is a sequence of commands. We consider particular programs P_n (for $n \geqslant 0$) in this question. The basic program P_0 just instructs the robot to move forward:

$$P_0 = \mathbf{F}.$$

The program P_{n+1} (for $n \geqslant 0$) involves performing P_n, turning left, performing P_n again, then turning right:

$$P_{n+1} = P_n\,\mathbf{L}\,P_n\,\mathbf{R}.$$

So, for example, $P_1 = \mathbf{F\,L\,F\,R}$.

(i) Write down the program P_2.

(ii) How far does the robot travel during the program P_n? In other words, how many **F** commands does it perform?

(iii) Let l_n be the total number of commands in P_n; so, for example, $l_0 = 1$ and $l_1 = 4$.

Write down an equation relating l_{n+1} to l_n. Hence write down a formula for l_n in terms of n. No proof is required. **Hint:** consider $l_n + 2$.

(iv) The robot starts at the origin, facing along the positive x-axis. What direction is the robot facing after performing the program P_n?

(v) The left-hand diagram on the opposite page shows the path the robot takes when it performs the program P_1. On the right-hand diagram opposite, draw the path it takes when it performs the program P_4.

(vi) Let (x_n, y_n) be the position of the robot after performing the program P_n, so $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 1)$. Give an equation relating (x_{n+1}, y_{n+1}) to (x_n, y_n).

What is (x_8, y_8)? What is (x_{8k}, y_{8k})?

Problem Solving Questions
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For answers, see (the MAT website)

2013, Question 5:

We define the *digit sum* of a non-negative integer to be the sum of its digits. For example, the digit sum of 123 is $1 + 2 + 3 = 6$.

(i) How many positive integers less than 100 have digit sum equal to 8?

Let n be a positive integer with $n < 10$.

(ii) How many positive integers less than 100 have digit sum equal to n?

(iii) How many positive integers less than 1000 have digit sum equal to n?

(iv) How many positive integers between 500 and 999 have digit sum equal to 8?

(v) How many positive integers less than 1000 have digit sum equal to 8, and one digit at least 5?

(vi) What is the total of the digit sums of the integers from 0 to 999 inclusive?

Problem Solving Questions
(From the Oxford MAT Tests)

For answers, see (the MAT website)

2014, Question 5:

Poets use *rhyme schemes* to describe which lines of a poem rhyme. Each line is denoted by a letter of the alphabet, with the same letter given to two lines that rhyme. To say that a poem has the rhyming scheme ABABCDED, indicates that the first and third lines rhyme, the second and fourth lines rhyme, and the sixth and eighth lines rhyme, but no others.

More precisely, the first line of the poem is given the letter A. If a subsequent line rhymes with an earlier line, it is given the same letter; otherwise, it is given the first unused letter. (For the purposes of this question, you can assume that we have an infinite supply of "letters", not just the 26 letters of the alphabet.)

The purpose of this question is to investigate how many different rhyme schemes there are for poems of n lines. We write r_n for this number.

(i) There are five different rhyming schemes for poems of three lines (so $r_3 = 5$). List them.

Let $c_{n,k}$ denote the number of rhyme schemes for poems with lines n that use exactly k different letters. For example $c_{3,2} = 3$ corresponding to the rhyming schemes AAB, ABA and ABB.

(ii) What is $c_{n,1}$ for $n \geqslant 1$?

What is $c_{n,n}$?

Explain your answers.

(iii) Suppose that $1 < k < n$. By considering the final letter of a rhyming scheme, explain why

$$c_{n,k} = kc_{n-1,k} + c_{n-1,k-1} \ .$$

(iv) Write down an equation showing how to calculate r_n in terms of the $c_{n,k}$. Hence calculate r_4.

(v) Give a formula for $c_{n,2}$ in terms of n (for $n \geqslant 2$). Justify your answer.