

Poets use *rhyme schemes* to describe which lines of a poem rhyme. Each line is denoted by a letter of the alphabet, with the same letter given to two lines that rhyme. To say that a poem has the rhyming scheme ABABCDEDED, indicates that the first and third lines rhyme, the second and fourth lines rhyme, and the sixth and eighth lines rhyme, but no others.

More precisely, the first line of the poem is given the letter A. If a subsequent line rhymes with an earlier line, it is given the same letter; otherwise, it is given the first unused letter. (For the purposes of this question, you can assume that we have an infinite supply of “letters”, not just the 26 letters of the alphabet.)

The purpose of this question is to investigate how many different rhyme schemes there are for poems of  $n$  lines. We write  $r_n$  for this number.

- (i) There are five different rhyming schemes for poems of three lines (so  $r_3 = 5$ ). List them.

Let  $c_{n,k}$  denote the number of rhyme schemes for poems with lines  $n$  that use exactly  $k$  different letters. For example  $c_{3,2} = 3$  corresponding to the rhyming schemes AAB, ABA and ABB.

- (ii) What is  $c_{n,1}$  for  $n \geq 1$ ?

What is  $c_{n,n}$ ?

Explain your answers.

- (iii) Suppose that  $1 < k < n$ . By considering the final letter of a rhyming scheme, explain why

$$c_{n,k} = kc_{n-1,k} + c_{n-1,k-1} .$$

- (iv) Write down an equation showing how to calculate  $r_n$  in terms of the  $c_{n,k}$ . Hence calculate  $r_4$ .

- (v) Give a formula for  $c_{n,2}$  in terms of  $n$  (for  $n \geq 2$ ). Justify your answer.