

The function $f(x)$ is defined for all real numbers and has the following properties, valid for all x and y :

- (A) $f(x + y) = f(x) f(y)$.
- (B) $df/dx = f(x)$.
- (C) $f(x) > 0$.

Throughout this question, these should be the only properties of f that you use; no marks will be awarded for any use of the exponential function.

Let $a = f(1)$.

(i) Show that $f(0) = 1$.

(ii) Let

$$I = \int_0^1 f(x) dx.$$

Show that $I = a - 1$.

(iii) The trapezium rule with n steps is used to produce an estimate I_n for the integral I . Show that

$$I_n = \frac{1}{2n} \left(\frac{b+1}{b-1} \right) (a-1)$$

where $b = f(1/n)$.

(iv) Given that $I_n \geq I$ for all n , show that

$$a \leq \left(1 + \frac{2}{2n-1} \right)^n.$$