

Amy and Brian play a game together, as follows. They take it in turns to write down a number from the set $\{0, 1, 2\}$, with Amy playing first. On each turn (except Amy's first turn), the player must not repeat the number just played by the previous player.

In their first version of the game, Brian wins if, after he plays, the sum of all the numbers played so far is a multiple of 3. For example, Brian will win after the sequence

2,0	1,2	1,0
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(where we draw a box around each round) because the sum of the numbers is 6. Amy wins if Brian has not won within five rounds; for example, Amy wins after the sequence

2,0	1,2	1,2	0,2	1,2
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- (i) Show that if Amy starts by playing either 1 or 2, then Brian can immediately win.
- (ii) Suppose, instead, Amy starts by playing 0. Show that Brian can always win within two rounds.

They now decide to change the rules so that Brian wins if, after he plays, the sum of all the numbers played so far is *one less than a multiple of 3*. Again, Amy wins if Brian has not won within five rounds. It is still the case that a player must not repeat the number just played previously.

- (iii) Show that if Amy starts by playing either 0 or 2, then Brian can immediately win.
- (iv) Suppose, instead, Amy starts by playing 1. Explain why it cannot benefit Brian to play 2, assuming Amy plays with the best strategy.
- (v) So suppose Amy starts by playing 1, and Brian then plays 0. How should Amy play next?
- (vi) Assuming both play with the best strategies, who will win the game? Explain your answer.