Suppose that x satisfies the equation

$$x^3 = 2x + 1. \qquad (*)$$

(i) Show that

$$x^4 = x + 2x^2 \qquad \text{and} \qquad x^5 = 2 + 4x + x^2.$$

(ii) For every integer $k \geqslant 0$, we can uniquely write

$$x^k = A_k + B_k x + C_k x^2$$

where A_k, B_k, C_k are integers. So, in part (i), it was shown that

$$A_4 = 0,\ B_4 = 1, C_4 = 2 \qquad \text{and} \qquad A_5 = 2,\ B_5 = 4,\ C_5 = 1.$$

Show that

$$A_{k+1} = C_k, \qquad B_{k+1} = A_k + 2C_k, \qquad C_{k+1} = B_k.$$

(iii) Let

$$D_k = A_k + C_k - B_k.$$

Show that $D_{k+1} = -D_k$ and hence that

$$A_k + C_k = B_k + (-1)^k.$$

(iv) Let $F_k = A_{k+1} + C_{k+1}$. Show that

$$F_k + F_{k+1} = F_{k+2}.$$