Consider sequences of the letters M, X and W. *Valid* sequences are made up according to the rule that an M and a W can never be adjacent in the sequence. So M, XMXW, and XMMXW are examples of valid sequences, whereas the sequences MW and XWMX are not valid.

(i) Clearly, there are 3 valid sequences of length 1. List all valid sequences of length 2.

(ii) Let $g(n)$ denote the number of valid sequences of length n. Further, let $m(n)$, $x(n)$, $w(n)$ denote the number of valid sequences of length n that start with an M, an X, a W respectively.

Explain why

$$\begin{aligned} m(n) &= w(n), \\ m(n) &= m(n-1) + x(n-1) \quad \text{for } n > 1, \\ x(n) &= 2m(n-1) + x(n-1) \quad \text{for } n > 1, \end{aligned}$$

and write down a formula for $g(n)$ in terms of $m(n)$ and $x(n)$.

Hence compute $g(3)$, and verify that $g(4) = 41$.

(iii) Given a sequence using these letters then we say that it is *reflexive* if the following operation on the sequence does not change it: reverse the letters in the sequence, and then replace each occurrence of M by W and vice versa. So MXW, WXXM and XWXMX are reflexive strings, but MXM and XMXX are not. Let $r(n)$ be the number of valid, reflexive sequences of length n.

If a sequence is reflexive and has odd length, what must the middle letter be? Explain your answer.

Hence, show that

$$r(n) = \begin{cases} x\left(\frac{n+1}{2}\right) & \text{if } n \text{ is odd,} \\ x\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \end{cases}$$