Various Questions
(From AEA Papers)

For answers, see the AEA website

2002, Question 6:

6. **Figure 2**

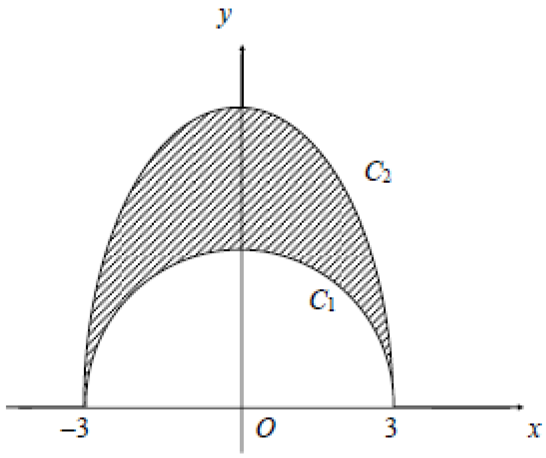

Figure 2 shows a sketch of part of two curves C_1 and C_2 for $y \geq 0$.

The equation of C_1 is $y = m_1 - x^{n_1}$ and the equation of C_2 is $y = m_2 - x^{n_2}$, where m_1, m_2, n_1 and n_2 are positive integers with $m_2 > m_1$.

Both C_1 and C_2 are symmetric about the line $x = 0$ and they both pass through the points (3, 0) and (–3, 0).

Given that $n_1 + n_2 = 12$, find

(*a*) the possible values of n_1 and n_2,

(4)

(*b*) the exact value of the smallest possible area between C_1 and C_2, simplifying your answer,

(8)

(*c*) the largest value of x for which the gradients of the two curves can be the same. Leave your answer in surd form.

(5)

2002, Question 7:

7. A student was attempting to prove that $x = \frac{1}{2}$ is the only real root of

$$x^3 + \tfrac{3}{4}x - \tfrac{1}{2} = 0.$$

The attempted solution was as follows.

$$x^3 + \tfrac{3}{4}x = \tfrac{1}{2}$$

$\therefore$ $$x(x^2 + \tfrac{3}{4}) = \tfrac{1}{2}$$

$\therefore$ $$x = \tfrac{1}{2}$$

or $$x^2 + \tfrac{3}{4} = \tfrac{1}{2}$$

i.e. $$x^2 = -\tfrac{1}{4} \quad \text{no solution}$$

$\therefore$ $$\text{only real root is } x = \tfrac{1}{2}$$

(*a*) Explain clearly the error in the above attempt.

(2)

(*b*) Give a correct proof that $x = \frac{1}{2}$ is the only real root of $x^3 + \frac{3}{4}x - \frac{1}{2} = 0$.

(3)

The equation

$$x^3 + \beta x - \alpha = 0 \qquad \text{(I)}$$

where α, β are real, $\alpha \neq 0$, has a real root at $x = \alpha$.

(*c*) Find and simplify an expression for β in terms of α and prove that α is the only real root provided $|\alpha| < 2$.

(6)

An examiner chooses a positive number α so that α is the only real root of equation (I) but the incorrect method used by the student produces 3 distinct real "roots".

(*d*) Find the range of possible values for α.

(7)

2004, Question 3:

3. $$f(x) = x^3 - (k + 4)x + 2k, \qquad \text{where } k \text{ is a constant.}$$

(a) Show that, for all values of k, the curve with equation $y = f(x)$ passes through the point (2, 0).

(1)

(b) Find the values of k for which the equation $f(x) = 0$ has exactly two distinct roots.

(5)

Given that $k > 0$, that the x-axis is a tangent to the curve with equation $y = f(x)$, and that the line $y = p$ intersects the curve in three distinct points,

(c) find the set of values that p can take.

(5)

2004, Question 6:

6. $$f(x) = x - [x], \ x \geq 0$$

where $[x]$ is the largest integer $\leq x$.

For example, $f(3.7) = 3.7 - 3 = 0.7$; $f(3) = 3 - 3 = 0$.

(a) Sketch the graph of $y = f(x)$ for $0 \leq x < 4$.

(3)

(b) Find the value of p for which $\int_2^p f(x)\,dx = 0.18$.

(3)

Given that

$$g(x) = \frac{1}{1+kx}, \quad x \geq 0, \quad k > 0,$$

and that $x_0 = \frac{1}{2}$ is a root of the equation $f(x) = g(x)$,

(c) find the value of k.

(2)

(d) Add a sketch of the graph of $y = g(x)$ to your answer to part (a).

(1)

The root of $f(x) = g(x)$ in the interval $n < x < n + 1$ is x_n, where n is an integer.

(e) Prove that

$$2x_n^2 - (2n - 1)x_n - (n + 1) = 0.$$

(4)

(f) Find the smallest value of n for which $x_n - n < 0.05$.

(4)

2005, Question 1:

1. A point P lies on the curve with equation

$$x^2 + y^2 - 6x + 8y = 24.$$

Find the greatest and least possible values of the length OP, where O is the origin.

(6)

2006, Question 4:

4. The line with equation $y = mx$ is a tangent to the circle C_1 with equation

$$(x + 4)^2 + (y - 7)^2 = 13.$$

(*a*) Show that m satisfies the equation

$$3m^2 + 56m + 36 = 0.$$

(4)

The tangents from the origin O to C_1 touch C_1 at the points A and B.

(*b*) Find the coordinates of the points A and B.

(8)

Another circle C_2 has equation $x^2 + y^2 = 13$. The tangents from the point $(4, -7)$ to C_2 touch it at the points P and Q.

(*c*) Find the coordinates of either the point P or the point Q.

(2)

2010, Question 6:

6. (a) Given that $x^4 + y^4 = 1$, prove that $x^2 + y^2$ is a maximum when $x = \pm y$, and find the maximum and minimum values of $x^2 + y^2$.

(7)

(b) On the same diagram, sketch the curves C_1 and C_2 with equations $x^4 + y^4 = 1$ and $x^2 + y^2 = 1$ respectively.

(2)

(c) Write down the equation of the circle C_3, centre the origin, which touches the curve C_1 at the points where $x = \pm y$.

(1)

2010, Question 7:

7.

$$f(x) = [1 + \cos(x + \tfrac{\pi}{4})][1 + \sin(x + \tfrac{\pi}{4})], \qquad 0 \leqslant x \leqslant 2\pi$$

(a) Show that $f(x)$ may be written in the form

$$f(x) = (\tfrac{1}{\sqrt{2}} + \cos x)^2, \qquad 0 \leqslant x \leqslant 2\pi$$

(5)

(b) Find the range of the function $f(x)$.

(2)

The graph of $y = f(x)$ is shown in Figure 2.

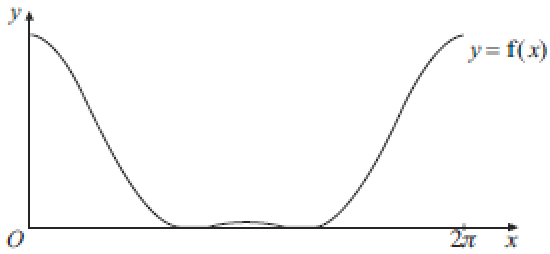

Figure 2

(c) Find the coordinates of all the maximum and minimum points on this curve.

(6)

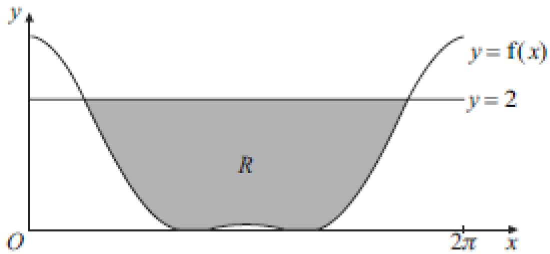

Figure 3

The region R, bounded by $y = 2$ and $y = f(x)$, is shown shaded in Figure 3.

(d) Find the area of R.

(8)

2014, Question 7:

7.

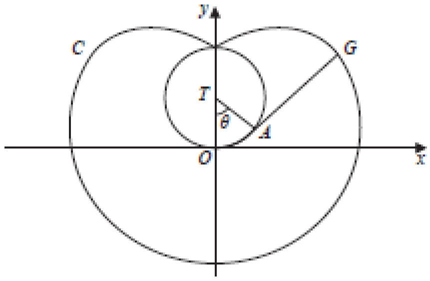

Figure 2

A circular tower stands in a large horizontal field of grass. A goat is attached to one end of a string and the other end of the string is attached to the fixed point O at the base of the tower. Taking the point O as the origin $(0, 0)$, the centre of the base of the tower is at the point $T(0, 1)$. The radius of the base of the tower is 1. The string has length π and you may ignore the size of the goat. The curve C represents the edge of the region that the goat can reach as shown in Figure 2.

(a) Write down the equation of C for $y < 0$.

(1)

When the goat is at the point $G(x, y)$, with $x > 0$ and $y > 0$, as shown in Figure 2, the string lies along OAG where OA is an arc of the circle with angle $OTA = \theta$ radians and AG is a tangent to the circle at A.

(b) With the aid of a suitable diagram show that

$$x = \sin\theta + (\pi - \theta)\cos\theta$$

$$y = 1 - \cos\theta + (\pi - \theta)\sin\theta$$

(5)

(c) By considering $\int y \frac{\mathrm{d}x}{\mathrm{d}\theta}\,\mathrm{d}\theta$, show that the area between C, the positive x-axis and the positive y-axis can be expressed in the form

$$\int_0^\pi u \sin u \,\mathrm{d}u + \int_0^\pi u^2 \sin^2 u \,\mathrm{d}u + \int_0^\pi u \sin u \cos u \,\mathrm{d}u$$

(5)

(d) Show that $\int_0^\pi u^2 \sin^2 u \,\mathrm{d}u = \frac{\pi^3}{6} + \int_0^\pi u \sin u \cos u \,\mathrm{d}u$

(4)

(e) Hence find the area of grass that can be reached by the goat.

(8)