Core 3 Integration Questions
(From AEA Papers)

For answers, see the AEA website

2003, Question 7:

7. **Figure 2**

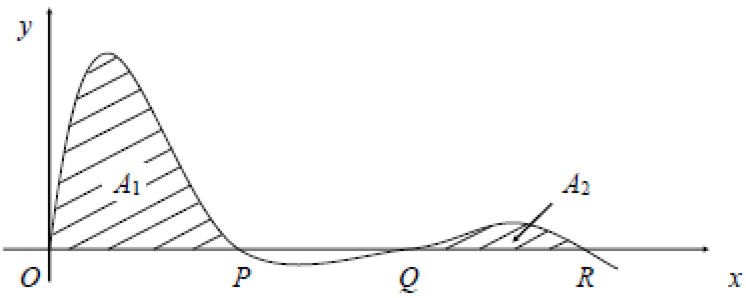

Figure 3 shows a sketch of part of the curve C with question

$$y = e^{-x} \sin x, \quad x \geq 0.$$

(*a*) Find the coordinates of the points P, Q and R where C cuts the positive axis.

(2)

(*b*) Use integration by parts to show that

$$\int e^{-x} \sin x \, dx = -\tfrac{1}{2} e^{-x} (\sin x + \cos x) + \text{constant}.$$

(5)

The terms of the sequence $A_1, A_2, \ldots, A_n, \ldots$ represent areas between C and the x-axis for successive portions of C where y is positive. The area represented by A_1 and A_2 are shown in Figure 3.

(*c*) Find an expression for A_n in terms of n and π.

(6)

(d) Show that $A_1 + A_2 + \ldots + A_n + \ldots$ is a geometric series with sum to infinity

$$\frac{e^{\pi}}{2(e^{\pi} - 1)}.$$

(5)

(*e*) Given that

$$\int_0^{\infty} e^{-x} \sin x \, dx = \tfrac{1}{2},$$

find the exact value of

$$\int_0^{\infty} \left| e^{-x} \sin x \right| \, dx$$

and simplify your answer.

(4)

2005, Question 7:

7. (*a*) Use the substitution $x = \sec\theta$ to show that

$$\int \sqrt{(x^2 - 1)} \, dx$$

can be written as

$$\int \sec\theta \tan^2\theta \, d\theta.$$

(3)

(*b*) Use integration by parts to show that

$$\int \sec\theta \tan^2\theta \, d\theta = \frac{1}{2}\left[\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + \text{constant}.$$

(7)

(*c*) Evaluate $\int_0^{\frac{\pi}{4}} \sin x \sqrt{(\cos 2x)} \, dx$.

(9)

2010, Question 5:

5.

$$I = \int \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x, \qquad x > 1$$

(a) Use the substitution $x = 1 + u^{-1}$ to show that

$$I = -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + c.$$

(7)

(b) Hence show that

$$\int_{\sec\alpha}^{\sec\beta} \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x = \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right), \qquad 0 < \alpha < \beta < \tfrac{\pi}{2}$$

(5)

2011, Question 2:

2. Given that

$$\int_0^{\frac{\pi}{2}} \left(1 + \tan\left[\tfrac{1}{2}x\right]\right)^2 \mathrm{d}x = a + \ln b$$

find the value of a and the value of b.

(Total 7 marks)

2013, Question 5:

5. In this question u and v are functions of x. Given that $\int \mathrm{u}\,\mathrm{d}x$, $\int \mathrm{v}\,\mathrm{d}x$ and $\int \mathrm{uv}\,\mathrm{d}x$ satisfy

$$\int \mathrm{uv}\,\mathrm{d}x = \left(\int \mathrm{u}\,\mathrm{d}x\right)\times\left(\int \mathrm{v}\,\mathrm{d}x\right) \qquad \mathrm{uv} \neq 0$$

(a) show that $1 = \dfrac{\int \mathrm{u}\,\mathrm{d}x}{\mathrm{u}} + \dfrac{\int \mathrm{v}\,\mathrm{d}x}{\mathrm{v}}$ **(3)**

Given also that $\dfrac{\int \mathrm{u}\,\mathrm{d}x}{\mathrm{u}} = \sin^2 x$,

(b) use part (a) to write down an expression, in terms of x, for $\dfrac{\int \mathrm{v}\,\mathrm{d}x}{\mathrm{v}}$, **(1)**

(c) show that

$$\frac{1}{\mathrm{u}}\frac{\mathrm{du}}{\mathrm{d}x} = \frac{1 - 2\sin x \cos x}{\sin^2 x}$$

(3)

(d) hence use integration to show that $\mathrm{u} = A\mathrm{e}^{-\cot x}\operatorname{cosec}^2 x$, where A is an arbitrary constant. **(6)**

(e) By differentiating $\mathrm{e}^{\tan x}$ find a similar expression for v. **(2)**

2013, Question 6:

6. (a) Starting from $[f(x) - \lambda g(x)]^2 \geqslant 0$ show that λ satisfies the quadratic inequality

$$\left(\int_a^b [g(x)]^2 \, dx\right)\lambda^2 - 2\left(\int_a^b f(x)g(x) \, dx\right)\lambda + \int_a^b [f(x)]^2 \, dx \geqslant 0$$

where a and b are constants and λ can take any real value.

(2)

(b) Hence prove that

$$\left[\int_a^b f(x)g(x) \, dx\right]^2 \leqslant \left[\int_a^b [f(x)]^2 \, dx\right] \times \left[\int_a^b [g(x)]^2 \, dx\right]$$

(3)

(c) By letting $f(x) = 1$ and $g(x) = \left(1 + x^3\right)^{\frac{1}{2}}$ show that

$$\int_{-1}^{2} \left(1 + x^3\right)^{\frac{1}{2}} \, dx \leqslant \frac{9}{2}$$

(4)

(d) Show that $\displaystyle\int_{-1}^{2} x^2 \left(1 + x^3\right)^{\frac{1}{2}} \, dx = \frac{12\sqrt{3}}{5}$

(3)

(e) Hence show that

$$\frac{144}{55} \leqslant \int_{-1}^{2} \left(1 + x^3\right)^{\frac{1}{2}} \, dx$$

(4)