6. (a) Starting from $[\mathrm{f}(x) - \lambda \mathrm{g}(x)]^2 \geqslant 0$ show that λ satisfies the quadratic inequality

$$\left(\int_a^b [\mathrm{g}(x)]^2 \,\mathrm{d}x\right)\lambda^2 - 2\left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x\right)\lambda + \int_a^b [\mathrm{f}(x)]^2 \,\mathrm{d}x \geqslant 0$$

where a and b are constants and λ can take any real value.

(2)

(b) Hence prove that

$$\left[\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x\right]^2 \leqslant \left[\int_a^b [\mathrm{f}(x)]^2 \,\mathrm{d}x\right] \times \left[\int_a^b [\mathrm{g}(x)]^2 \,\mathrm{d}x\right]$$

(3)

(c) By letting $\mathrm{f}(x) = 1$ and $\mathrm{g}(x) = \left(1 + x^3\right)^{\frac{1}{4}}$ show that

$$\int_{-1}^{2} \left(1 + x^3\right)^{\frac{1}{2}} \,\mathrm{d}x \leqslant \frac{9}{2}$$

(4)

(d) Show that $\int_{-1}^{2} x^2 \left(1 + x^3\right)^{\frac{1}{4}} \,\mathrm{d}x = \frac{12\sqrt{3}}{5}$

(3)

(e) Hence show that

$$\frac{144}{55} \leqslant \int_{-1}^{2} \left(1 + x^3\right)^{\frac{1}{2}} \,\mathrm{d}x$$

(4)