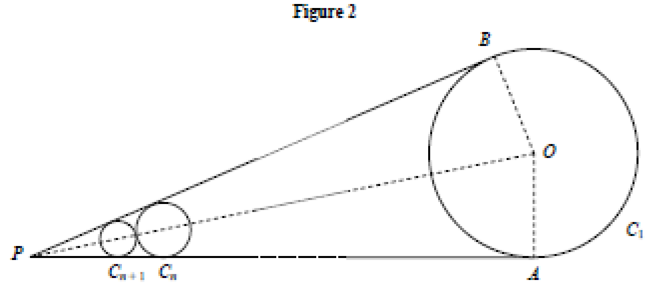
7.

**Figure 2**




The circle $C_1$ has centre $O$ and radius $R$. The tangents $AP$ and $BP$ to $C_1$ meet at the point $P$ and angle $APB = 2\alpha$, $0 < \alpha < \frac{\pi}{2}$. A sequence of circles $C_1, C_2, \ldots, C_n, \ldots$ is drawn so that each new circle $C_{n+1}$ touches each of $C_n$, $AP$ and $BP$ for $n = 1, 2, 3, \ldots$ as shown in Figure 2. The centre of each circle lies on the line $OP$.

(*a*) Show that the radii of the circles form a geometric sequence with common ratio

$$\frac{1-\sin\alpha}{1+\sin\alpha}.$$

**(5)**

(*b*) Find, in terms of $R$ and $\alpha$, the total area enclosed by all the circles, simplifying your answer.

**(3)**

The area inside the quadrilateral $PAOB$, not enclosed by part of $C_1$ or any of the other circles, is $S$.

(*c*) Show that

$$S = R^2\left(\alpha + \cot\alpha - \frac{\pi}{4}\operatorname{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right).$$

**(5)**

(*d*) Show that, as $\alpha$ varies,

$$\frac{dS}{d\alpha} = R^2\cot^2\alpha\left(\frac{\pi}{4}\cos\alpha - 1\right).$$

**(4)**

(*e*) Find, in terms of $R$, the least value of $S$ for $\frac{\pi}{6} \le \alpha \le \frac{\pi}{4}$.

**(3)**