7. A student was attempting to prove that $x = \frac{1}{2}$ is the only real root of

$$x^3 + \tfrac{3}{4}x - \tfrac{1}{2} = 0.$$

The attempted solution was as follows.

$$x^3 + \tfrac{3}{4}x = \tfrac{1}{2}$$

$\therefore$ $$x(x^2 + \tfrac{3}{4}) = \tfrac{1}{2}$$

$\therefore$ $$x = \tfrac{1}{2}$$

or $$x^2 + \tfrac{3}{4} = \tfrac{1}{2}$$

i.e. $$x^2 = -\tfrac{1}{4} \quad \text{no solution}$$

$\therefore$ $$\text{only real root is } x = \tfrac{1}{2}$$

(*a*) Explain clearly the error in the above attempt.

(2)

(*b*) Give a correct proof that $x = \frac{1}{2}$ is the only real root of $x^3 + \frac{3}{4}x - \frac{1}{2} = 0$.

(3)

The equation

$$x^3 + \beta x - \alpha = 0 \qquad \text{(I)}$$

where α, β are real, $\alpha \neq 0$, has a real root at $x = \alpha$.

(*c*) Find and simplify an expression for β in terms of α and prove that α is the only real root provided $|\alpha| < 2$.

(6)

An examiner chooses a positive number α so that α is the only real root of equation (I) but the incorrect method used by the student produces 3 distinct real "roots".

(*d*) Find the range of possible values for α.

(7)