

Harmonic Form

$$a \sin x \pm b \cos x = R \sin(x \pm \alpha)$$

$$a \cos x \pm b \sin x = R \cos(x \mp \alpha)$$

Where

$$R = \sqrt{a^2 + b^2}$$

$$R \cos \alpha = a \quad \text{and} \quad R \sin \alpha = b$$

$$\Rightarrow \cos \alpha = \frac{a}{R}, \quad \sin \alpha = \frac{b}{R}, \quad \tan \alpha = \frac{b}{a}$$

Harmonic Form Teaching Activity

1. Draw $y = 3\sin x + 2\cos x$. Make observations.
2. Experiment with other versions of $y = a\sin x + b\cos x$ making further observations.
3. Realize that $y = 3\sin x + 2\cos x$ can be written in the form $y = R\sin(x + \alpha)$ and try to suggest reasons for these values ($R \approx 3.6$, $\alpha \approx 0.58$).
4. Do the algebra. Equating both expressions and using double angle formulae:

$$3\sin x + 2\cos x = R\sin(x + \alpha)$$
$$3\sin x + 2\cos x = R\sin x \cos \alpha + R\cos x \sin \alpha$$

Therefore:

$$3\sin x = R\sin x \cos \alpha \Rightarrow 3 = R\cos \alpha \Rightarrow \cos \alpha = \frac{3}{R}$$

$$2\cos x = R\cos x \sin \alpha \Rightarrow 2 = R\sin \alpha \Rightarrow \sin \alpha = \frac{2}{R}$$

$$R = \sqrt{3^2 + 2^2} \quad \text{and} \quad \tan \alpha = \frac{2}{3}$$

5. Textbook or exam questions where students convert equations into harmonic form.
6. Some harder questions...

1. Find R and α where $7\cos x + 5\sin x = R\sin(x + \alpha)$
2. Find the coordinate of the minimum point in the interval $0 \leq x \leq 360$ of $4\sin x + 2\cos x$
3. Solve $12\sin \theta + 5\cos \theta = \frac{13}{2}$ for $\{\theta \in R: 0 \leq \theta \leq 2\pi\}$.