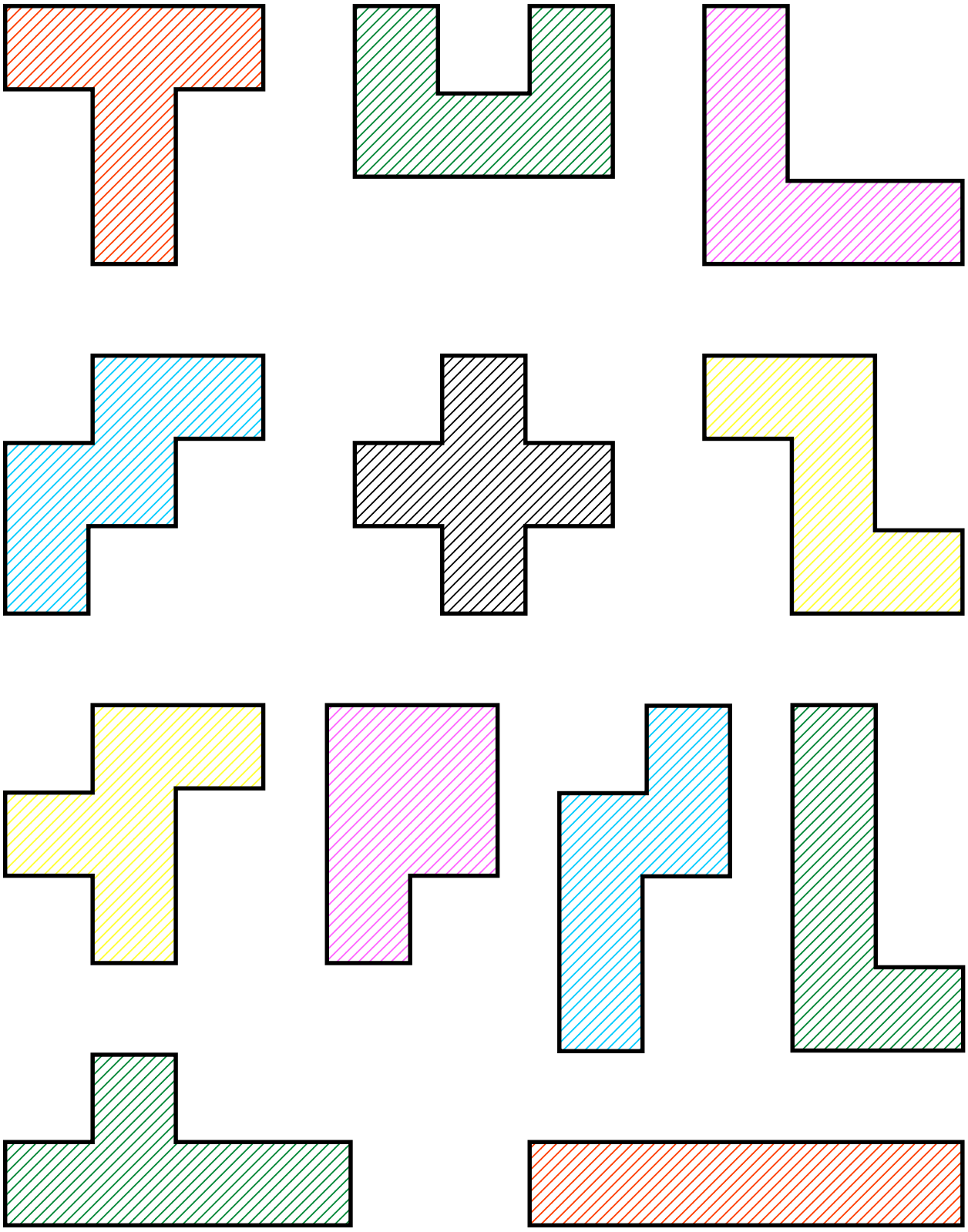
# Pentominoes



# Notes ~ 1

## Background

This unit is concerned with providing plenty of spatial work within a particular context. It could justifiably be titled "Puzzling with Pentominoes".

**Pentominoes** are just one subset of the set of Polyominoes.

**Polyominoes** is the general name given to plane shapes made by joining squares together.

Each type of polyomino is named according to how many squares are used to make it. So, there are monominoes (1 square only), dominoes (2 squares), triominoes (3 squares), tetrominoes (4 squares), pentominoes (5 squares), hexominoes (6 squares) and so on.

Though the idea of such shapes has been around in Recreational Mathematics since the beginning of the 1900's, it was not until the latter half of that century that they became really well-known.

In 1953 Solomon W Golomb (an American professor of mathematics) first introduced them and named them (derived from the idea of dominoes) together with some ideas of their possiblities for creating puzzles. They were taken up those interested but were little known outside of mathematical circles.

They were not brought to the notice of the world in general until 1957 when Martin Garden (in his famous column in the Scientific American) wrote about them, and they have remained a rich source of spatial recreations ever since.

**Pentominoes** (made from 5 squares) are the type of polyomino most worked with. There are 12 in the set and this means thay they are few enough to be handleable, yet quite enough to provide diversity.

How this material is used will depend upon each teacher's own inclinations and associated environment. It could be a major project or a few 'one-offs'. It could be done by individuals or by groups. It could be a background activity or an intense piece of work aimed at producing an exhibition for an open-evening. Whatever, it is usually enjoyed by most.

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# Notes ~ 2

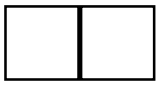
## Getting Started

A very good way of starting this work off is with a small investigation that might be introduced in this way.

> We wish to make some shapes using only squares, putting them together under certain conditions:
>
> - Only flat (2-dimensional) shapes are to be made.
> - Squares must join each other along the full length of an edge.
> - All the shapes made must differ from one another.
> - Two shapes are not considered to be different if one can be fitted exactly on top of the other, turning it over or around if necessary.

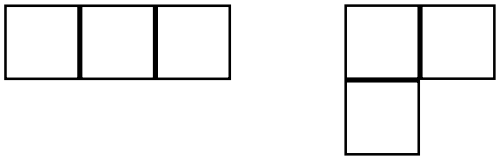
We start exploring the possiblities and, sooner or later, get organized and produce results along these lines, always seeing how many different shapes are possible each time.

Using **2** squares, yields only **1** possibility



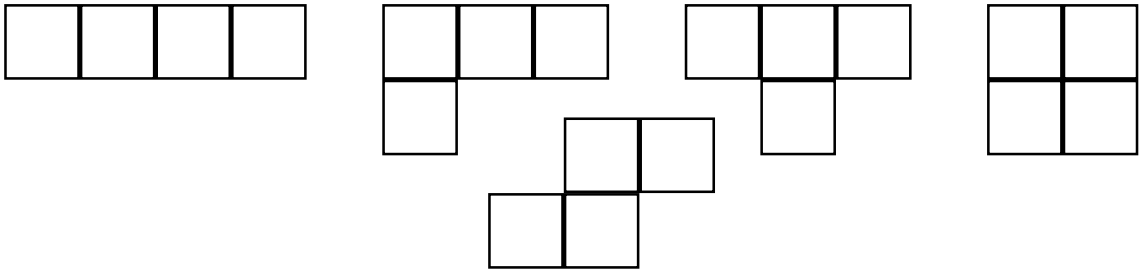
this is a domino (duo = 2)

Using **3** squares, gives **2** shapes



these are the trominoes (tri = 2)

Using **4** squares, gives **5** shapes



these are the tetrominoes (tetra = 4)

Using **5** squares, gives **12** shapes

(illustrated elsewhere)

these are the pentominoes (pent = 5)

Using **6** squares, gives **35** shapes

(not illustrated)

these are the hexominoes (hex = 6)

The number of possible shapes climbs quickly after that and there is no known formula by which the number of polyominoes can be found for any given number of squares. However, work on that continues. In the meantime, computer programs have been written to count the possibilities and definite values are now known for all cases containing up to 24 squares. The value for 24 squares is over 650 billion.

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# Notes ~ 3

Having found the pentominoes it is then a matter of making a set to work with. However this is done, they should be cut from card to facilitate handling. So card with a grid of squares marked on it helps things considerably. In a similar way paper with a grid of squares marked on it is useful when working on problems. Know then that master grids of varying sizes (5, 7, 10, 15 and 20 mm) are available via the ***trol*** menu under *Lined Grids*. Clearly the choice of size for the unit square used should be consistent throughout. 10 or 15 mm is recommended for those with a normal adult-level of dexterity, moving to 20 mm for less nimble fingers.

Alternatively, master copies of sets of pentominoes are available (based on 10, 15 and 20 mm unit-squares) to allow the sets to be printed directly on card and so avoid the necessity for any drawing.

## Further points

- Card used should be in a variety of colours to make individual sets as distinctive as possible, and pupils should put their initials on each shape. Using incorrect sets with duplicated and/or missing pieces can make many of the problems either much easier or impossible.
- When cutting the shapes out, it is recommended that the cut be made **inside** the line. This makes it a little easier to handle them when manoeuvring within the confines of a pre-drawn shape which is based on the same size unit-square.
- For solving problems of the space-filling variety, it is best to draw the required shape on squared-paper and place the pieces within the drawing. In any case, squared-paper will be needed for drawing the solutions.
- Identifying the pieces is essential, and the generally accepted way of doing this is used here in this unit. There is an ohp master showing the necessary letters. A little diversion could be provided by first asking what letters could be used, or perhaps giving the 12 letters and asking for the requisite matches to be made.

## Sources of further information

The standard work on the subject is

**Polyominoes, puzzles, patterns, problems and packings.**

Simon W Golomb

Princeton University Books 1994 ISBN 0-691-08573-0

*This is not only a complete book on the subject, but also contains a very large list of other published sources about Polyominoes and similar sorts of ideas.*

For even more information (as well as getting later results on some problems) look on the Web. For example, giving the search-engine

**www.google.com**

the entry

**pentominoes**

will find more than 3,000 possibilities, some excellent, some useless. You just have to look and choose those that suit you.

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# Notes ~ 4

## Further activities

- Analyse the shapes. What symmetries do they have? How many 'corners' does each have (convex and concave counted separately)? How many 'joins', that is where squares touch edge to edge? What are their perimeters? What relationships exist between these values?

- Invent a configuration and, given only the outline, invite others to solve it.

- Play a 'covering game'. Two players. Use one set of pentominoes and a suitable board made of unit-squares. An 8 by 8 is a good size. Players take turns placing a pentomino to cover squares. Pentominoes must 'fit' the squares and may not overlap any already placed. Player who is last able to place a piece wins. Strategies?

- How many pentominoes would there be in the set if, when looking for different shapes, it was not permitted to turn them over to match? This extended set is known as the one-sided pentominoes, and has not been investigated very much.

- Investigate the hexominoes. As there are so many (35) it is best to deal with a subset using some criteria to sort them out.

- It is not only squares that can be used as a basis for making shapes. Equilateral triangles are another profitable line of investigations. Use isometric grids, also available from the ***trol*** menu under 'Dotted Grids'. For starters, try 6 triangles, these are the hexiamonds.

- **Shapes in 3 dimensions**. If the pentominoes are made using cubes instead of squares, then it becomes possible to work with 3-dimensional shapes. Note that these are only 2-dimensional pentominoes which have been given a 'thickness' of 1 unit. It does not mean that 5 cubes are assembled in 3 dimensions, as this would admit several more shapes. (These are the pentacubes and there are 29 of them.)

  Having made them, the simplest problem then is to put all 12 together to make a cuboid. Clearly it will have a volume of 60 cubes, and it can be done as a 3 by 4 by 5**;** a 2 by 5 by 6**;** a 2 by 3 by 10. All of them are possible.

  8 of these 'solid' pentominoes can be assembled to make a twice-size replica of almost any one of them. It is NOT possible for the **I**, **T**, **W** and **X**.

  There are many other interesting shapes which can be set as puzzles to be copied.

  Since there is only likely to be one set of these available, this is an activity more suited to say a Maths Club.

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# Some Notes on the Problems ~ 1

**Problems 1 and 2** are the simplest of space-filling problems and serve as an introduction to all that follows, especially with regard to getting familiar with identifying the pieces. Since solutions can be found in abundance, a relatively easy task is to require a display to be made up.

**Problem 3** is a little more difficult than the previous two but could also be used to launch an investigation. How many different 5 by 5 squares can be found? This could be an on-going investigation, or competition, involving the whole class with results being drawn and posted-up as they are found. It does require a record-keeping system and/or notation that will allow repeats to be spotted. Note that in some cases the same five pentominoes can be put together in two different ways. Also note that 5 pieces can be chosen from 12 in 792 different ways, but that not all of them can be assembled to make a square!

**Problems 4 and 5** are a little harder and are also presented in a different way.

**Problem 6** offers a change and could well be set as a challenge to be done outside the classroom. It could be pointed out that the example offered could be improved considerably by simply altering the positions of a few of the pentominoes. For example, reflecting **V** and **W** about a horizontal line produces an increase of 11 squares, while moving the **U** outwards give another 5, and there are several other gainful changes to be found.
The maximum possible area of a field is 126 squares.
An extension of this activity is to impose either one (but not both) of these restrictions**:**

- the inside of the field must be a rectangle
- the outer perimeter must be in the shape of a rectangle.

**Problem 7** Something different again. The **X** pentomino is the easiest one to see will do it, but most people actually need to 'bend up' some of the pentominoes to see if they work. Only four pentominoes will NOT provide the correct net.

**Problem 8** is a good one to take away and do - once it has been checked that the requirement has been properly grasped.

**Problem 9** Before embarking on this it is necessary that what constitutes a tessellation is clearly understood - emphasis on pattern, repetition and no holes.
A suitable follow-on would be to find all the different ways in which the **F** pentomino will tessellate. The unit on Tessellations (also to be found from the **trol** menu) has a sheet which gives some idea of what is required.

**Problem 10** Since it says that two cannot be done, and one is given as an example, only nine remain to be found. This could be given to a group, requiring them to make all nine solutions and present them in the form of a poster.

**Problem 11** Could be done in a similar manner to the previous one, but now all twelve can be done including a *different* solution to the **V**.

**Problems 12 to 15** Bigger and bigger rectangles to be filled.

**Problem 16** This one could be solved by using the Problem 4 example plus the required solution. As an extra, ask for the 15 by 4 rectangle to be found.

**Problem 17** is a hard one which is why a partial solution is offered.
If more help is needed, offer (one at a time) **I P U X** to go on the left-hand side of those already provided.

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# Some Notes on the Problems ~ 2

**Problem 18** introduces the idea of shapes with 'holes' in them.
In this case the hole can be in any one of ten different positions (ignoring symmetrical variations) and it could be asked, of a group say, to find solutions to all ten possible cases.

**Problems 19 and 20** are not too difficult since the shapes are given.

**Problems 21 to 23** though similar in form to the previous two, are harder because the shape that has to be made is not given.
A much harder extra is to divide the full set of pentominoes into three groups of four and make each group into the same shape. It is unlikely to solved if merely stated in that form.
The first hint is that one group is **Y N L P**
The second hint is that another group is **F W T I**
But even then it is still not an easy problem.

**Problems 24 to 43** 20 problems. Share them out.

**Making Models** Not problems, just pleasurable activities making the two models offfered here.
They are the tetrahedron and the cube. Print on white card to allow for colouring-in, they are nothing without that.
Establishing just how the correct arrangement of the pentominoes was found is probably beyond the scope of most classrooms, though it is easier to see with the tetrahedron than the cube. On the cube it is worth looking closely at what has happened to the **W** pentomino and convince yourself it is correct.
As a challenge, look at Problem 36. Cut off the bottom six squares and it is the net of a cube. That means losing one complete pentomino and one square off another, or losing one square off each of six pentominoes. The challenge is to design a 'fudge' that will, at first glance, pass muster.

**Solutions** No solutions are provided, except for a few to the problem of making a rectangle from the complete set. Three (10 by 6, 12 by 5 and 15 by 4) are drawn large on a single sheet which could be used to make an ohp transparency. They are only given in outline here but would benefit from being coloured-in. Apart from being solutions, perhaps only needed for 'encouragement', they can be used for another activity. Expose one of the drawings for say 2 seconds (count elephants) and then cover it over. During the exposure period, pupils may not do anything but only look at the screen. During any time the drawing is NOT on view, pupils should attempt to re-create it on their desks using their own sets. How many exposures are needed before the first copy is completed? (Or all copies?)

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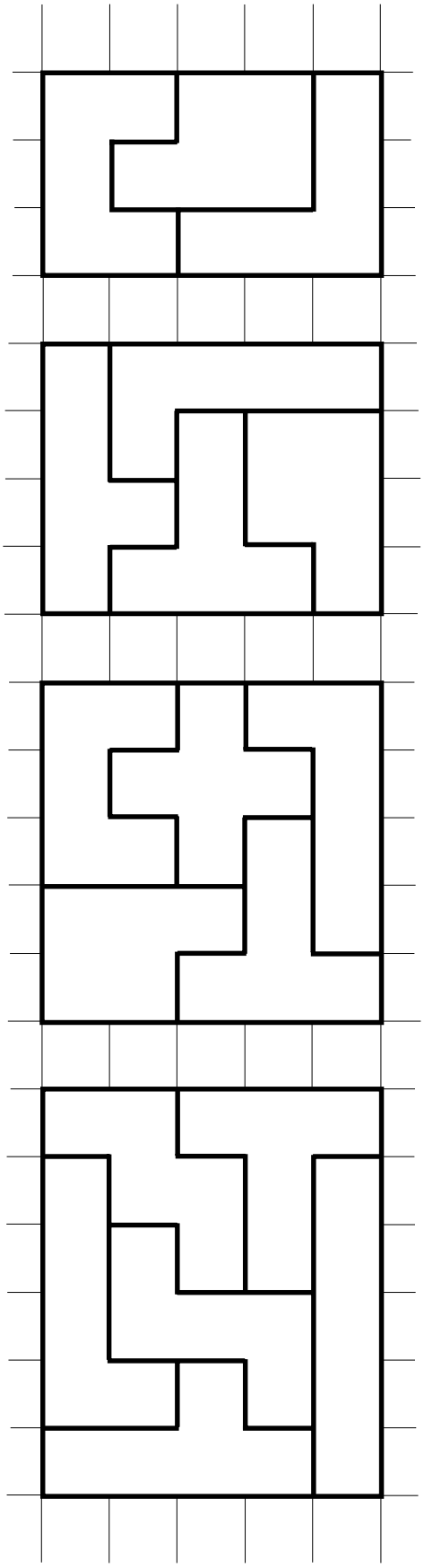
# Pentomino Problems 1 ~ 5



**1.** On the right is shown how the 3 pentominoes **P, U** and **V** can be fitted together to fill a rectangle which covers an area of 15 (5 by 3) squares.

Find **at least** two other ways of filling a rectangle of the same size with 3 pentominoes

**2.** On the right is shown how the 4 pentominoes **L, P, T** and **Y** can be fitted together to fill a rectangle which covers an area of 20 (5 by 4) squares.

Find **at least** two other ways of filling a rectangle of the same size with 4 pentominoes

**3.** On the right is shown how the 5 pentominoes **L, P, T, U** and **X** can be fitted together to fill a rectangle which covers an area of 25 (5 by 5) squares.

Find **at least** two other ways of filling a rectangle of the same size with 5 pentominoes

**4.** On the right is shown how 6 pentominoes can be fitted together to fill a 5 by 6 rectangle.

Use the **other** 6 pentominoes from the full set of 12, to fill another 5 by 6 rectangle.

**5.** Use the **I, N, V, W, Y** and **Z** pentominoes to make a 5 by 7 rectangle.
Then use the **other** 5 pentominoes from the full set of 12, to make a 5 by 5 rectangle.

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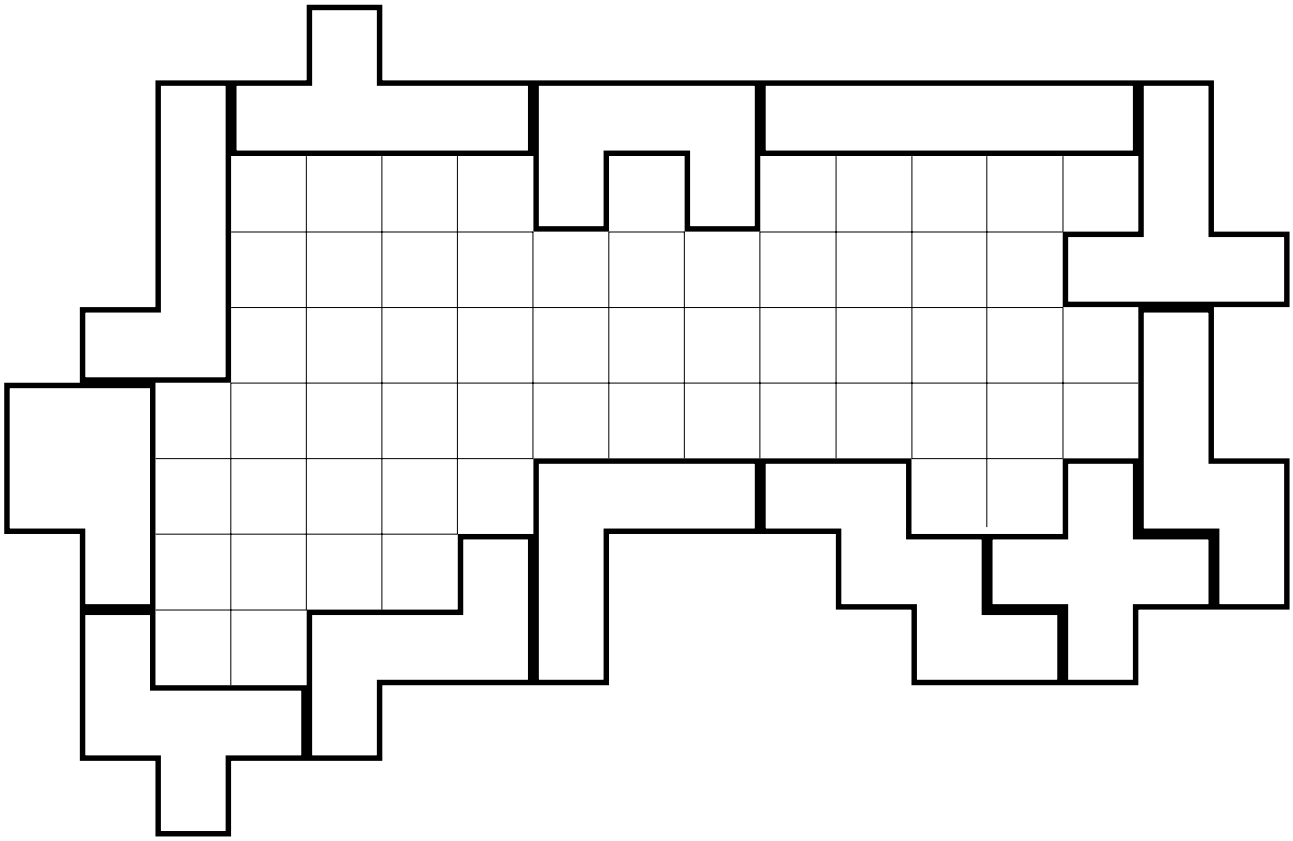
# Pentomino Problem 6

**6.** The drawing below shows the full set of 12 pentominoes arranged to enclose a ‘field’.
Notice that the pentominoes must touch along the full edge of a square and not just at the corners.



It can be seen (by counting) that the field encloses an area of 59 squares.
How many squares can you enclose inside a pentomino field?
You can grade your attempts by reference to this table.

| **Area** | **Grade** |
|---|---|
| under 70 | not trying! |
| 71 - 80 | E |
| 81 - 90 | D |
| 91 - 100 | C |
| 101 - 110 | B |
| 111 - 120 | A |
| over 120 | Super! |

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# Pentomino Problems 7 ~ 11

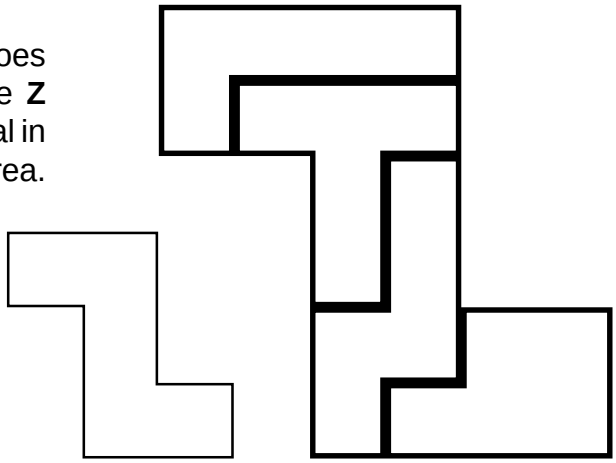
**7.** From the complete set of 12 pentominoes, which ones would be suitable to use as drawings of a net which would fold up to make an open-topped box. That is, a box without a lid.
*You will have to ignore the need for flaps to hold it together.*

**8.** It is possible to fill a 5 by 10 rectangle using 10 copies of the **Y** pentomino.
Make a drawing to show how this can be done.
*The drawing should be coloured so that the separate pentominoes can be seen.*

**9.** Any one of the 12 pentominoes can be used as the basic shape for drawing a tessellation. That is, using the same shape over and over again so as to fill up a flat space without leaving any holes. It is easier with some than with others.
The easiest are the **I**, **L**, **N**, **P**, **V**, **W**, **Y** and **Z**.
The **F**, **T**, **U** and **X** are a little more difficult and, if you are not careful, you will soon find holes appearing. Make a drawing to show how one these will tessellate.
You need to make the drawing big enough so it is easy to see how the pattern repeats. That probably means drawing at least 20 copies of the shape you are using. The careful use of colour helps to show how the pattern repeats itself.

**10.** The drawings on the right show how 4 pentominoes have been put together to make a copy of the **Z** pentomino. The copy is twice as big as the original in every direction which means it is four times the area.

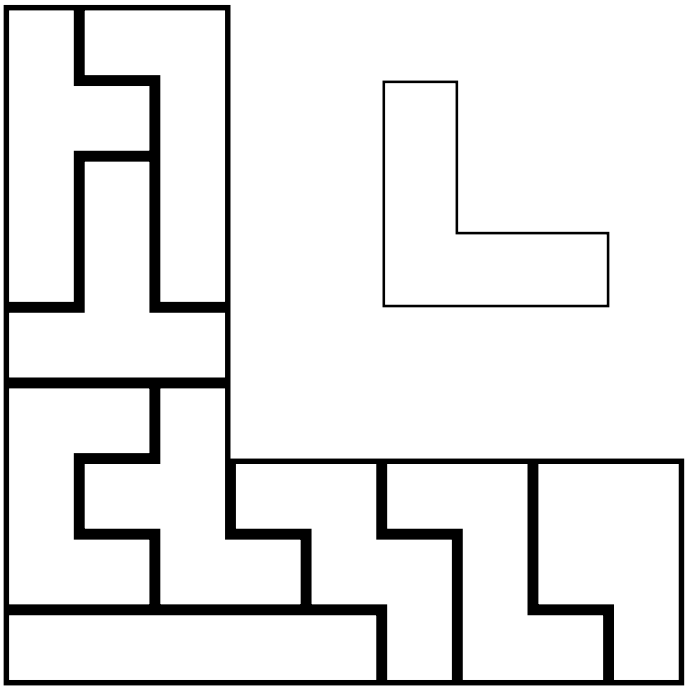
Choose any other pentomino (except the **V** or **X** for which this cannot be done) and make a copy of it twice as big using the other pentominoes.



**11.** From the complete set of pentominoes choose any one and set it aside.

Then, from the remaining 11, use any 9 to make a copy which is 3 times bigger than the one first chosen. (Note it is 9 times bigger in area.)

The example shown on the right is for the **V** pentomino.



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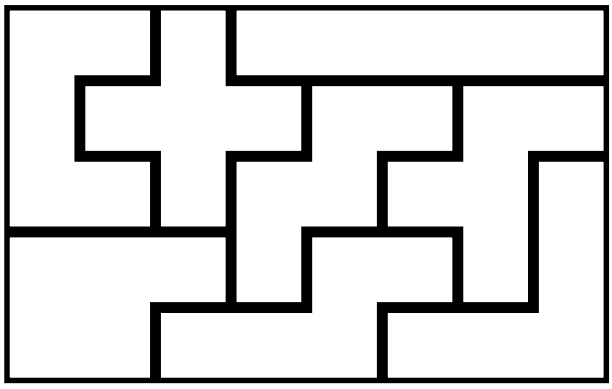
# Pentomino Problems 12 ~ 17

Any single pentomino covers an area of 5 squares. So the **I** pentomino by itself will fill a rectangle (1 by 5) having an area of 5 squares.
A rectangle, made of pentominoes, having an area of 10 squares must be 2 by 5 in size, but it is not difficult to see that no two (different) pentominoes can be found that can be put together to make a rectangle of that size.
The next smallest rectangle that can be made is the 3 by 5 with an area of 15 squares.
Then there is the 4 by 5, the 5 by 5, the 6 by 5 and the 7 by 5, and this covers all the possible rectangles up to an area of 35 squares.

**12.** When we get to an area of 40 squares, for the first time two differently sized rectangles become possible. It can be done with an 8 by 5 or a 4 by 10. Here is one possibility, from among many, for the 8 by 5



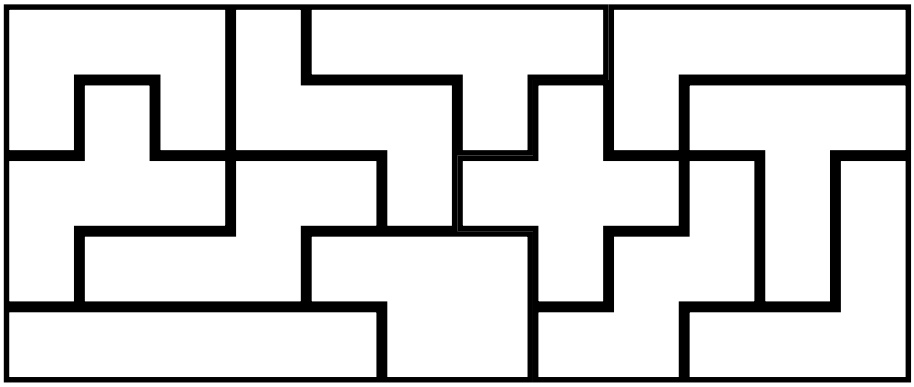
Make a drawing to show how it can be done for the 4 by 10.

**13.** A rectangle having an area of 45 squares can be made as a 3 by 15 or a 9 by 5. The first would be a solution to the triplication of the **I** pentomino (in Problem 11).
Find, and draw, a solution to the second.

**14.** Make a rectangle with pentominoes which has an area of 50 squares.

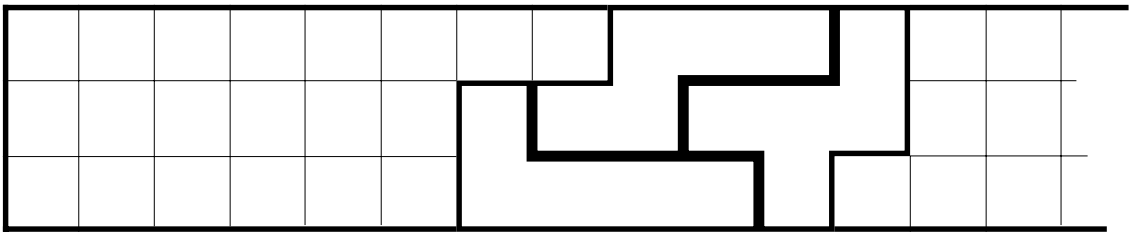
**15.** Make a rectangle with 11 pentominoes.

The usual puzzle with pentominoes requires a rectangle to be made which uses all 12 of the pieces. Such a rectangle clearly must have an area of 60 squares. It can be done in four different ways: as a 12 by 5, as a 10 by 6, as a 15 by 4, as a 20 by 3.
One solution, (to the 12 by 5) from many thousands, is shown on the right.



**16.** Find a solution to the 10 by 6 rectangle.

**17.** Whilst many of these problems can have thousands of solutions, the 20 by 3 rectangle can only be made in three different ways. The start of one is shown below.
Find, and draw, a complete solution.



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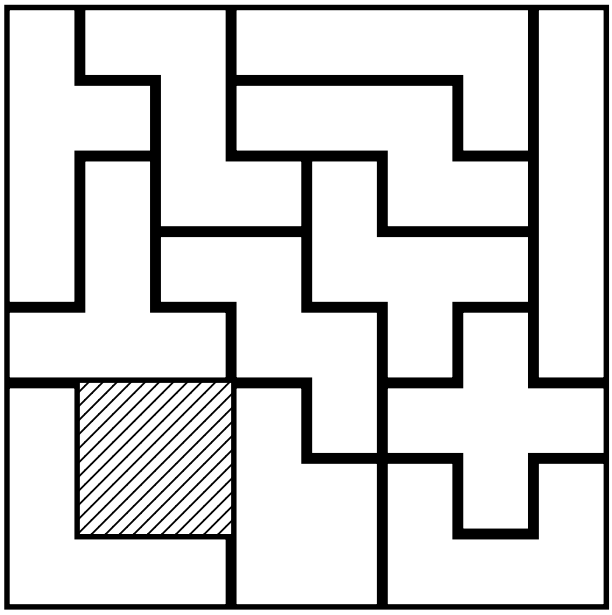
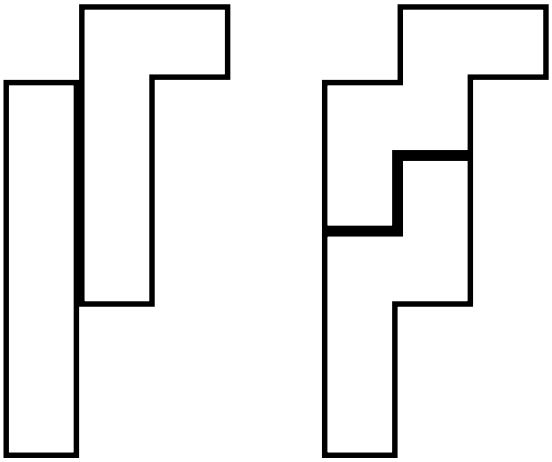
# Pentomino Problems 18 ~ 23

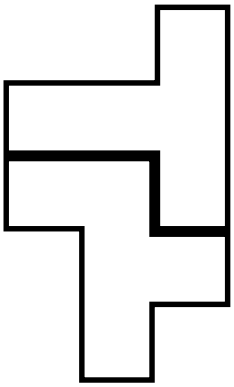
The normal-sized chess-board is made up of 64 squares arranged in an 8 by 8.
Since there are only 60 squares in a full set of pentominoes it would not be possible to use them to cover a complete chess-board. However, it is porrible to cover a chess-board provided a 'hole' of 4 squares (in a 2 by 2) is left in it. An example is shown on the right.
A solution is possible no matter where that hole is placed. (The hole is the shaded square)



**18.** Find a solution to covering the chess-board with the hole placed exactly in the middle of the board.

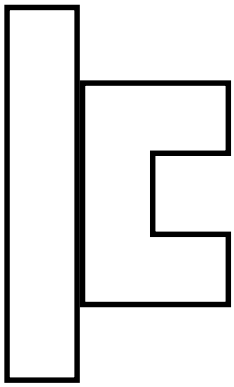


It is not too difficult to fit two pentominoes together to make a total shape and then choose two other pentominoes which fit together to make the **same** total shape.
This has been done in the drawings on the left, where it can be seen that the **I** and **L** have first been put together to make one total shape, and then the **W** and **N** have been used to make the same total shape.



**19.** Fit the **Z** and **P** pentominoes together to make the same total shape as the **F** and **T** shown on the left

**20.** Find two pentominoes which can be fitted together to make the same total shape as the **I** and **U** shown on the right



**21.** Put the **V** and **Z** pentominoes together to make the same total shape as the **L** and **N**

**22.** Put the **W** and **X** pentominoes together to make the same total shape as the **P** and **Y**

**23.** Put the **V** and **X** pentominoes together to make the same total shape as the **U** and **Y**
*Note that this shape has a 'hole' in it.*

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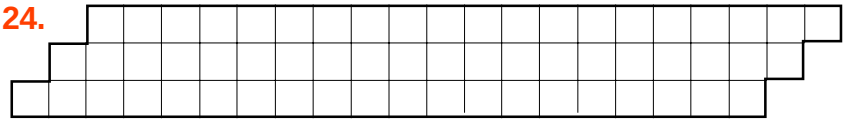
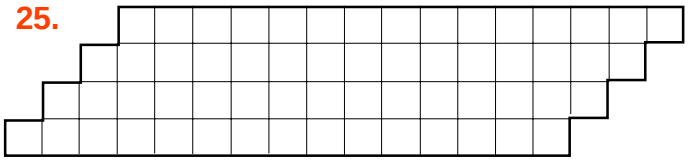
# Pentomino Problems 24 ~ 35

Each of the shapes shown on this sheet can be made
using a complete set of the 12 pentominoes.
*(The black squares represent holes.)*

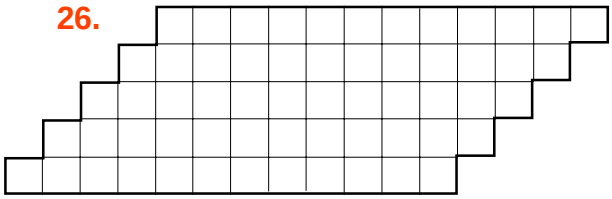
**24.**



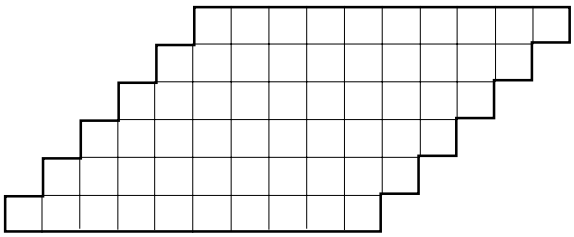
**25.**



**26.**



**27.**



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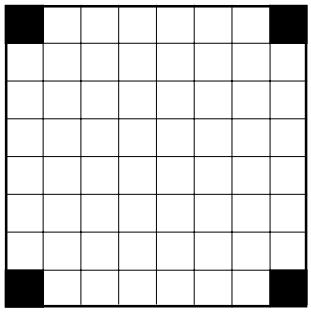

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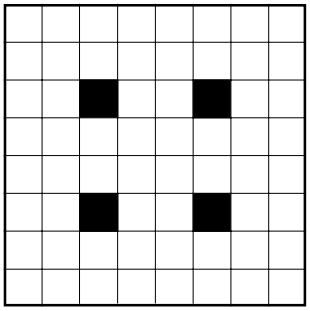

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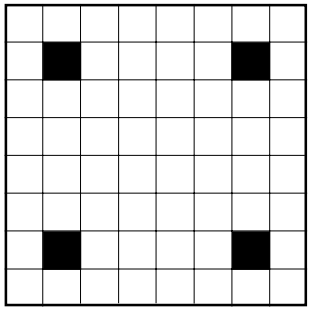

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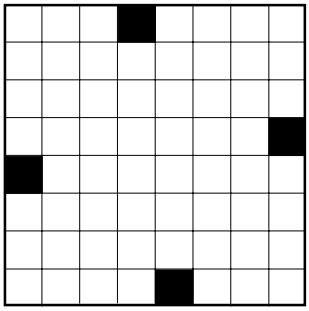

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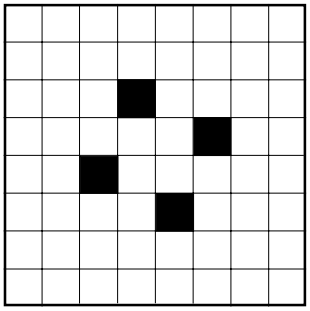

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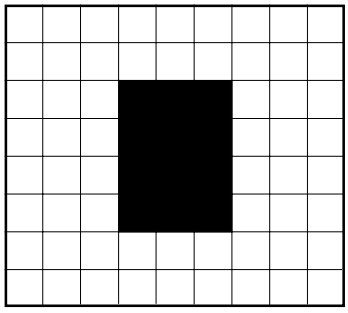

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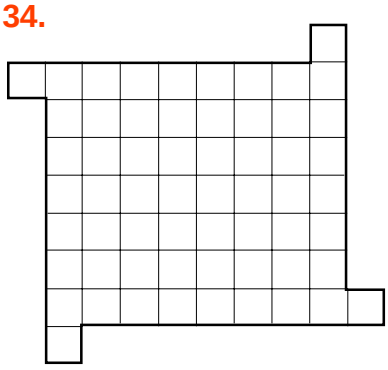

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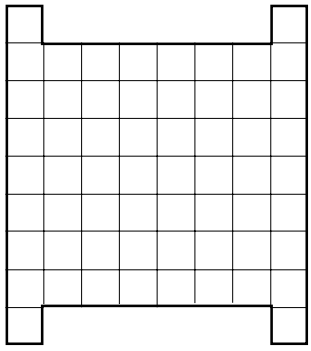

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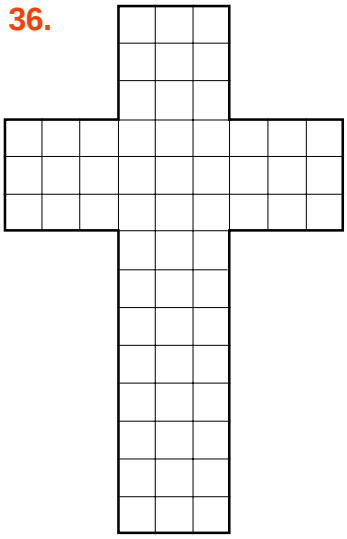
# Pentomino Problems 36 ~ 43

Each of the shapes shown on this sheet can be made
using a complete set of the 12 pentominoes.
*(The black squares represent holes.)*

**36.**



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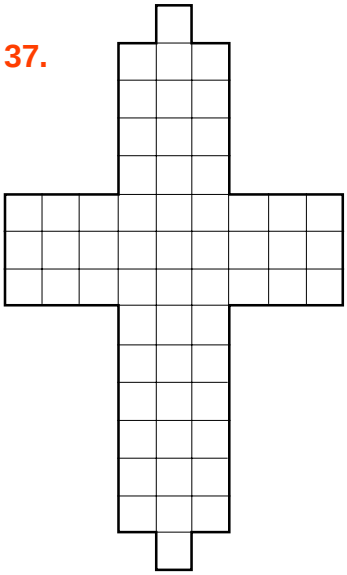

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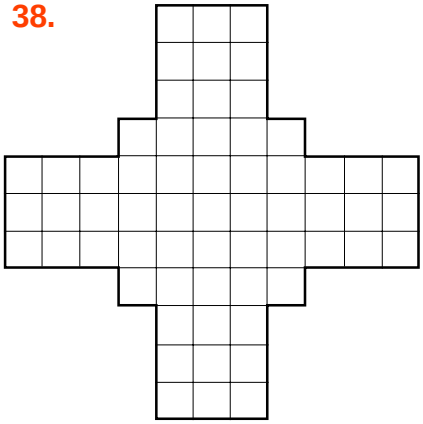

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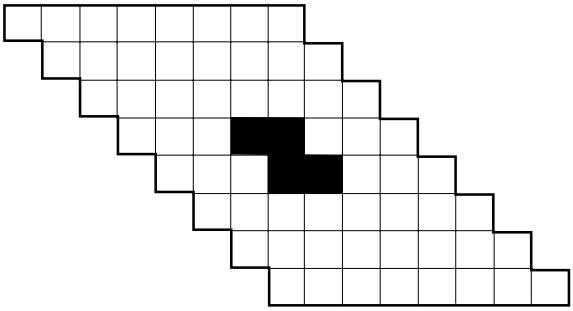

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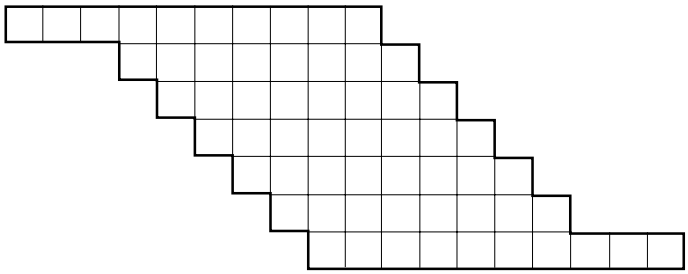

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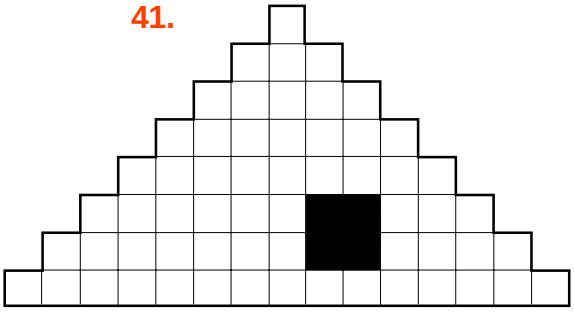

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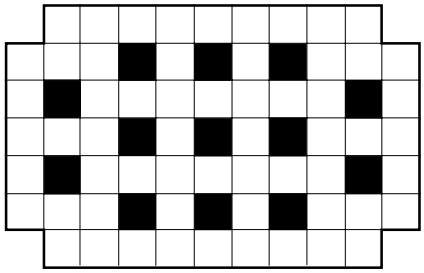
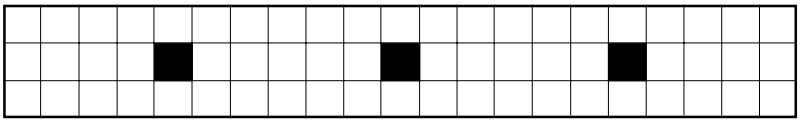


**43.**



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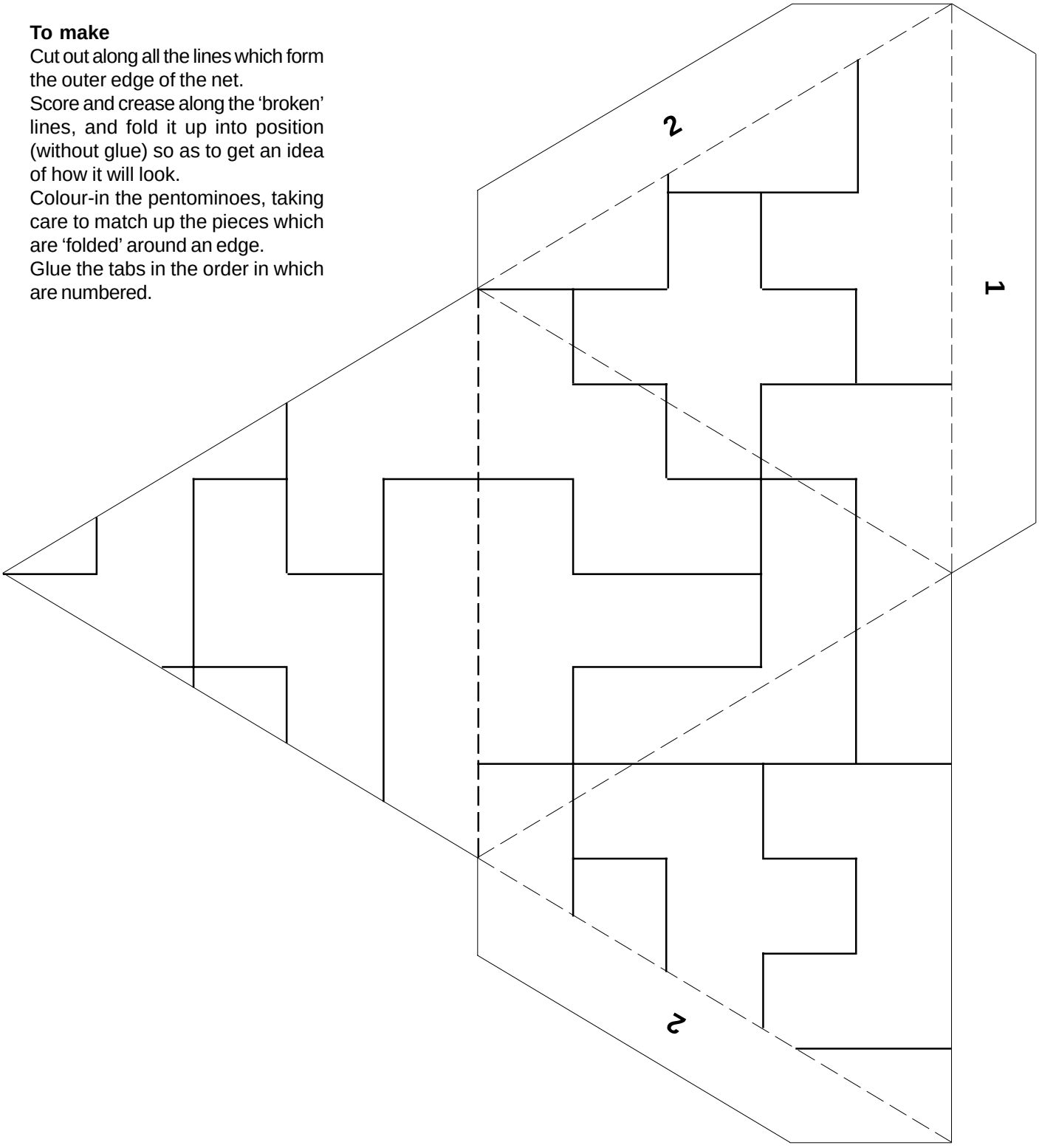
# Pentominoes on a Regular Tetrahedron

This net will make a regular tetrahedron which has been decorated over its 4 faces with the 12 pieces of a complete set of pentominoes.

**To make**

Cut out along all the lines which form the outer edge of the net.
Score and crease along the 'broken' lines, and fold it up into position (without glue) so as to get an idea of how it will look.
Colour-in the pentominoes, taking care to match up the pieces which are 'folded' around an edge.
Glue the tabs in the order in which are numbered.

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# Pentominoes on a Cube

This net will make a cube which has been decorated over its 6 faces with the 12 pieces of a complete set of pentominoes.

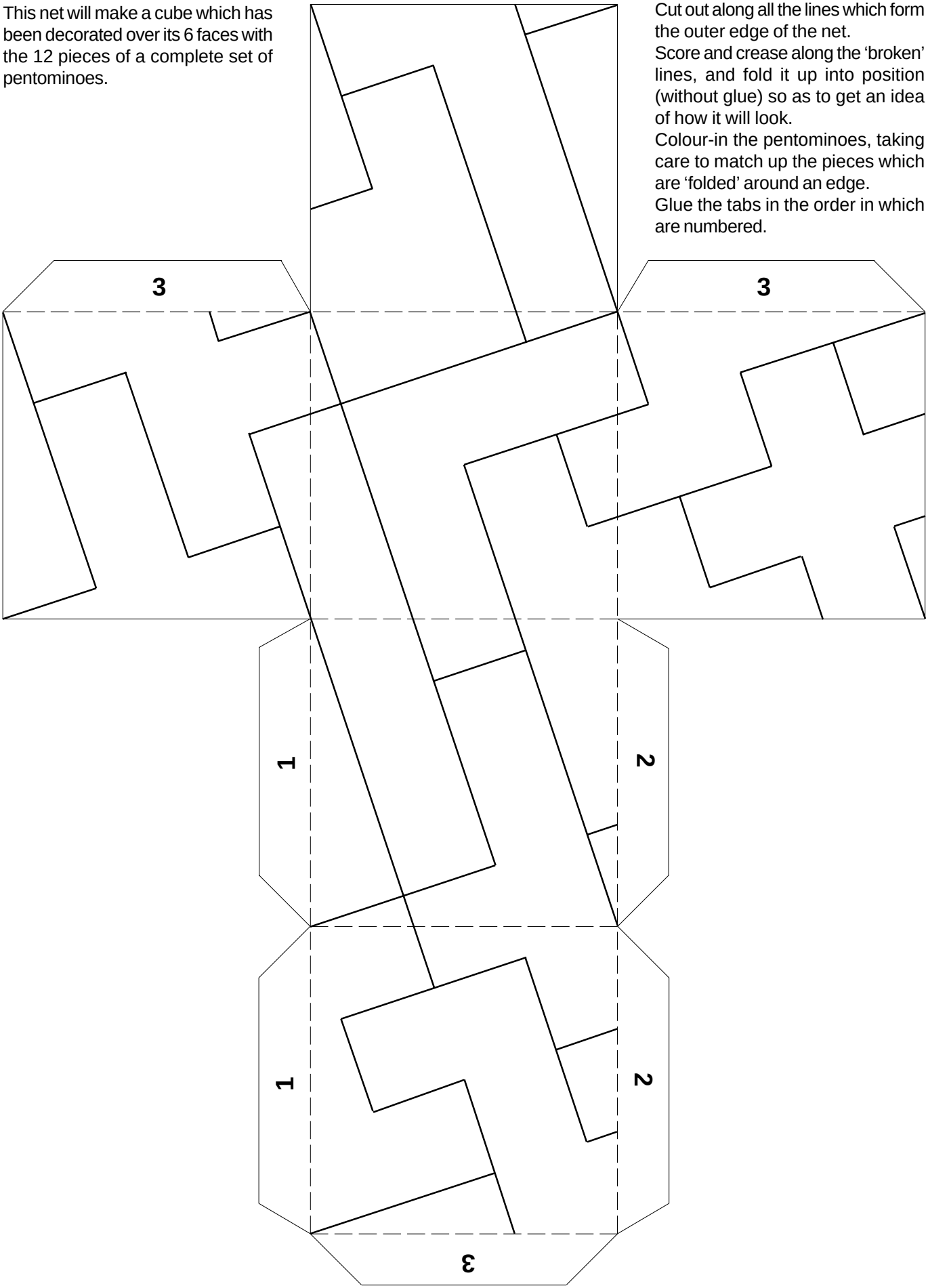
**To make**

Cut out along all the lines which form the outer edge of the net.

Score and crease along the 'broken' lines, and fold it up into position (without glue) so as to get an idea of how it will look.

Colour-in the pentominoes, taking care to match up the pieces which are 'folded' around an edge.

Glue the tabs in the order in which are numbered.

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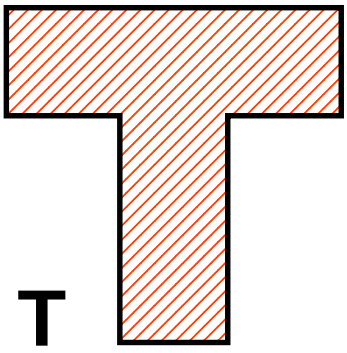
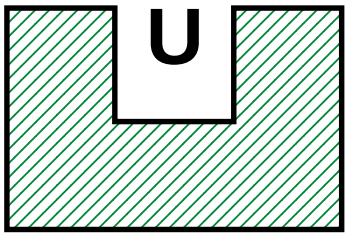
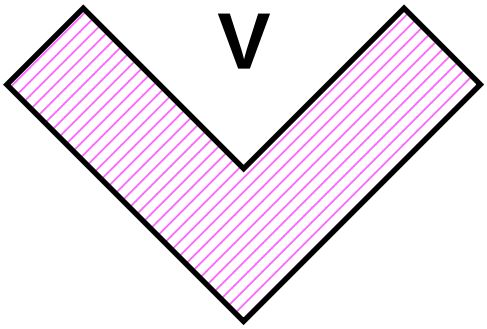
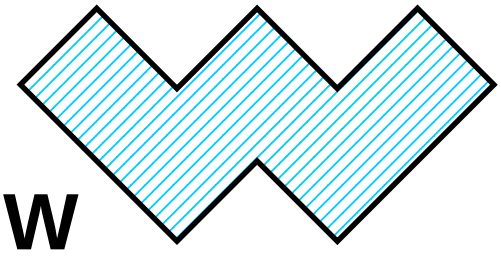
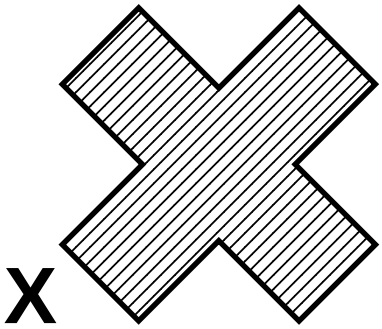
# Pentomino Identification

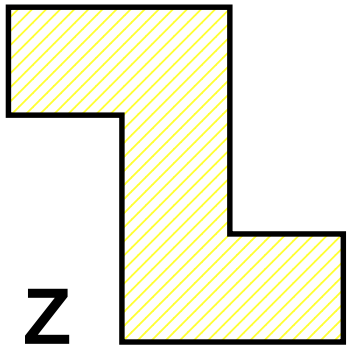


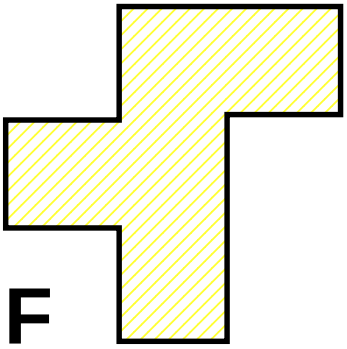



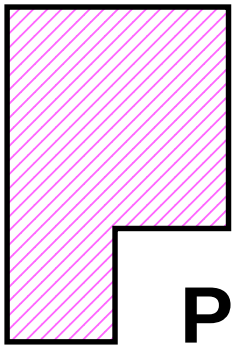



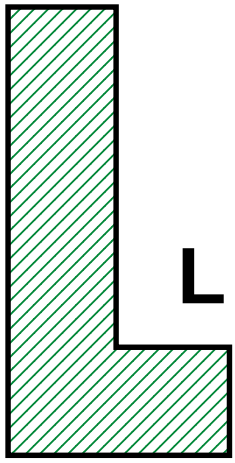



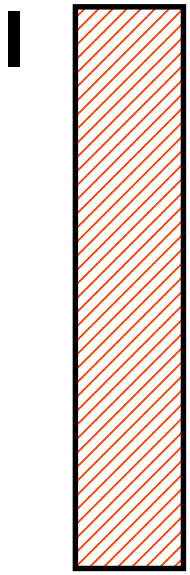



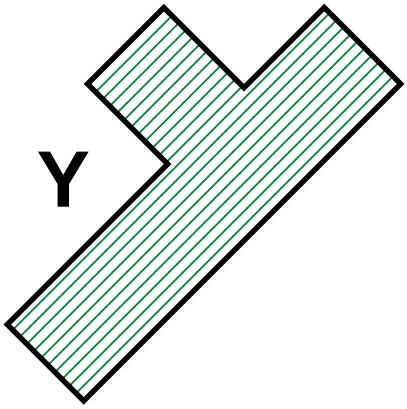



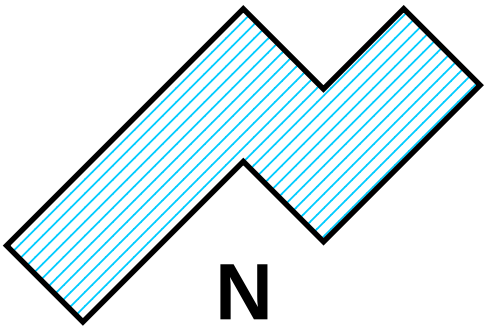

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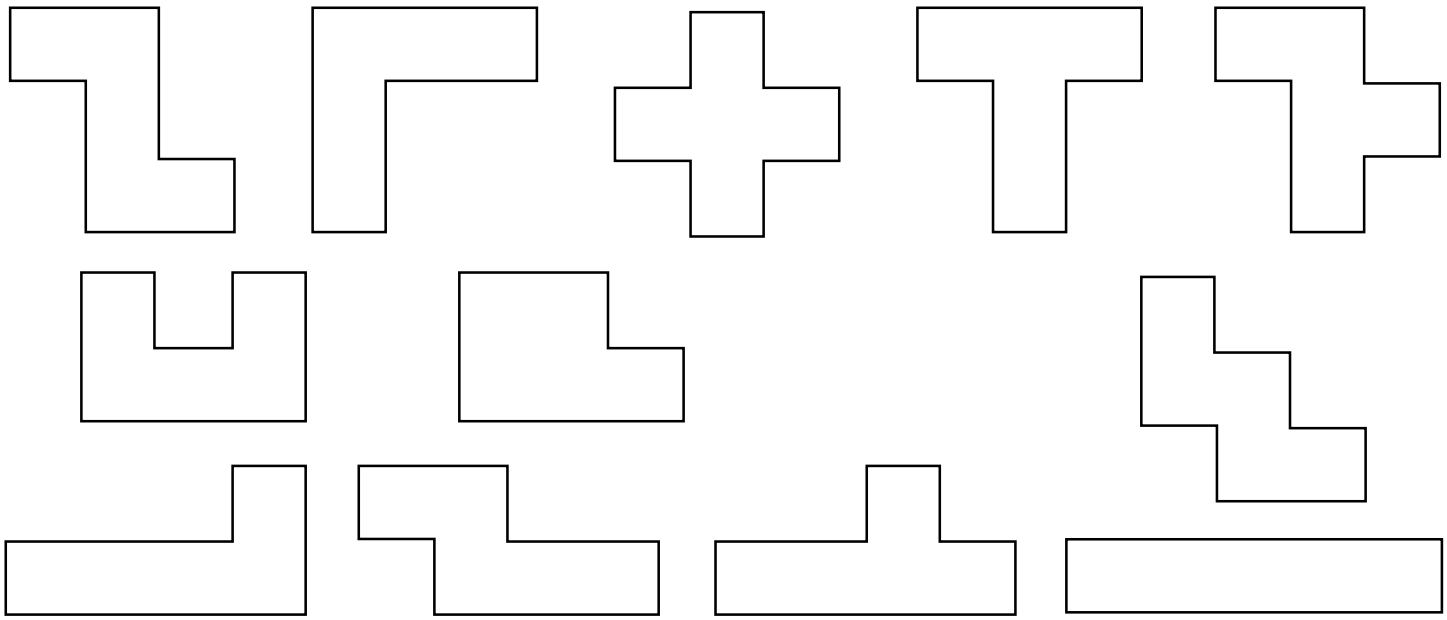
10 mm



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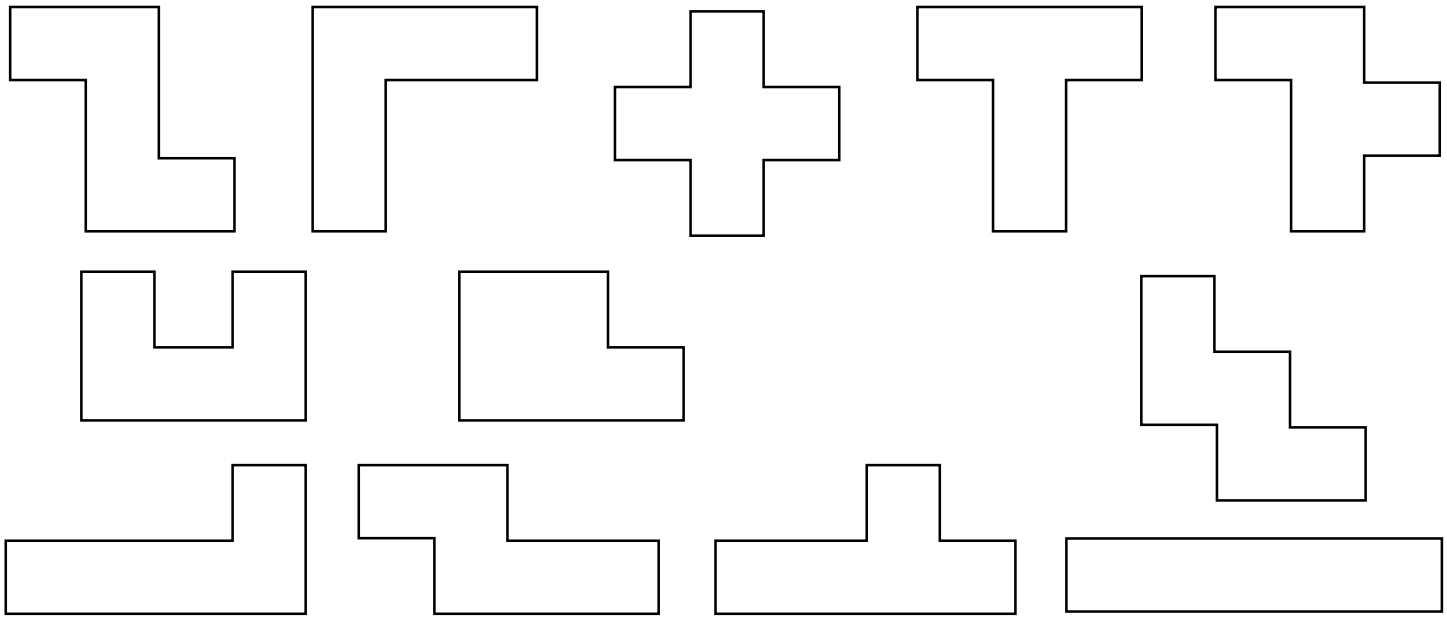

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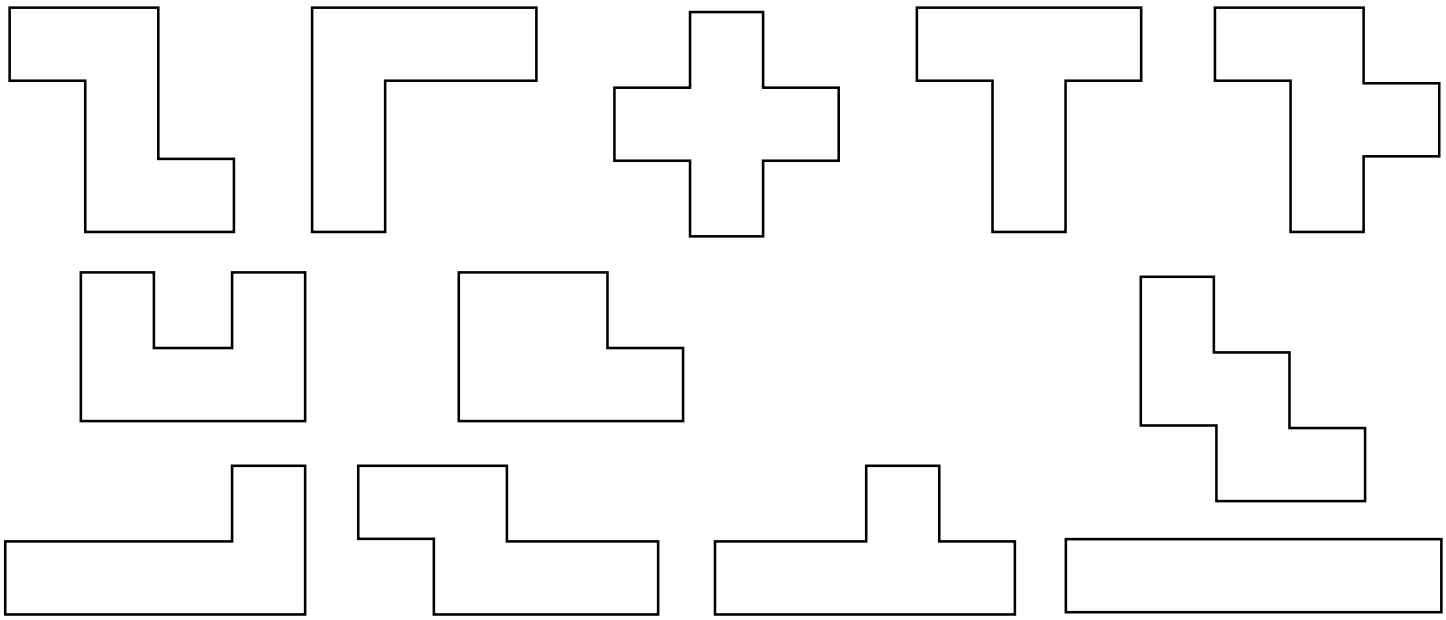

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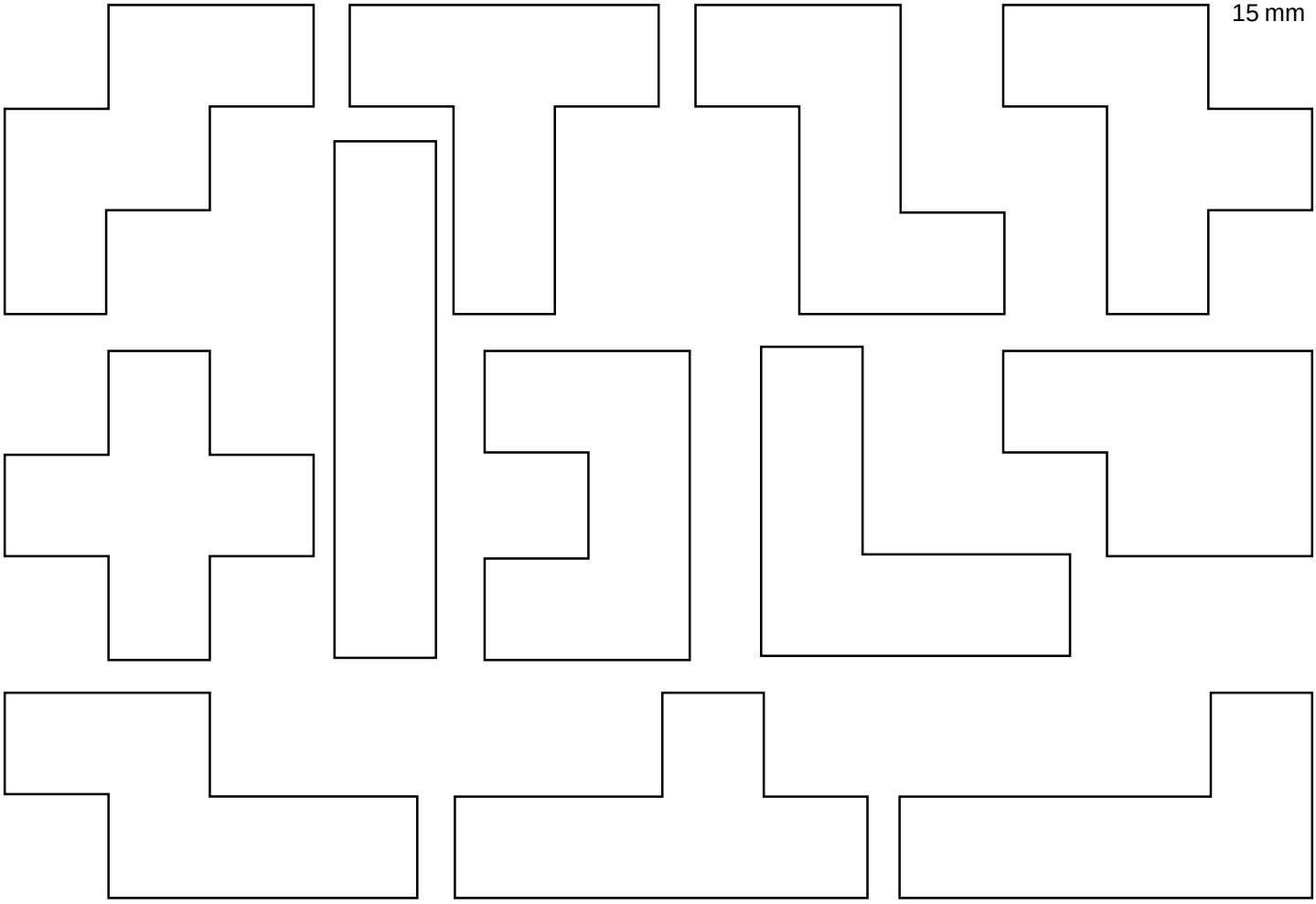


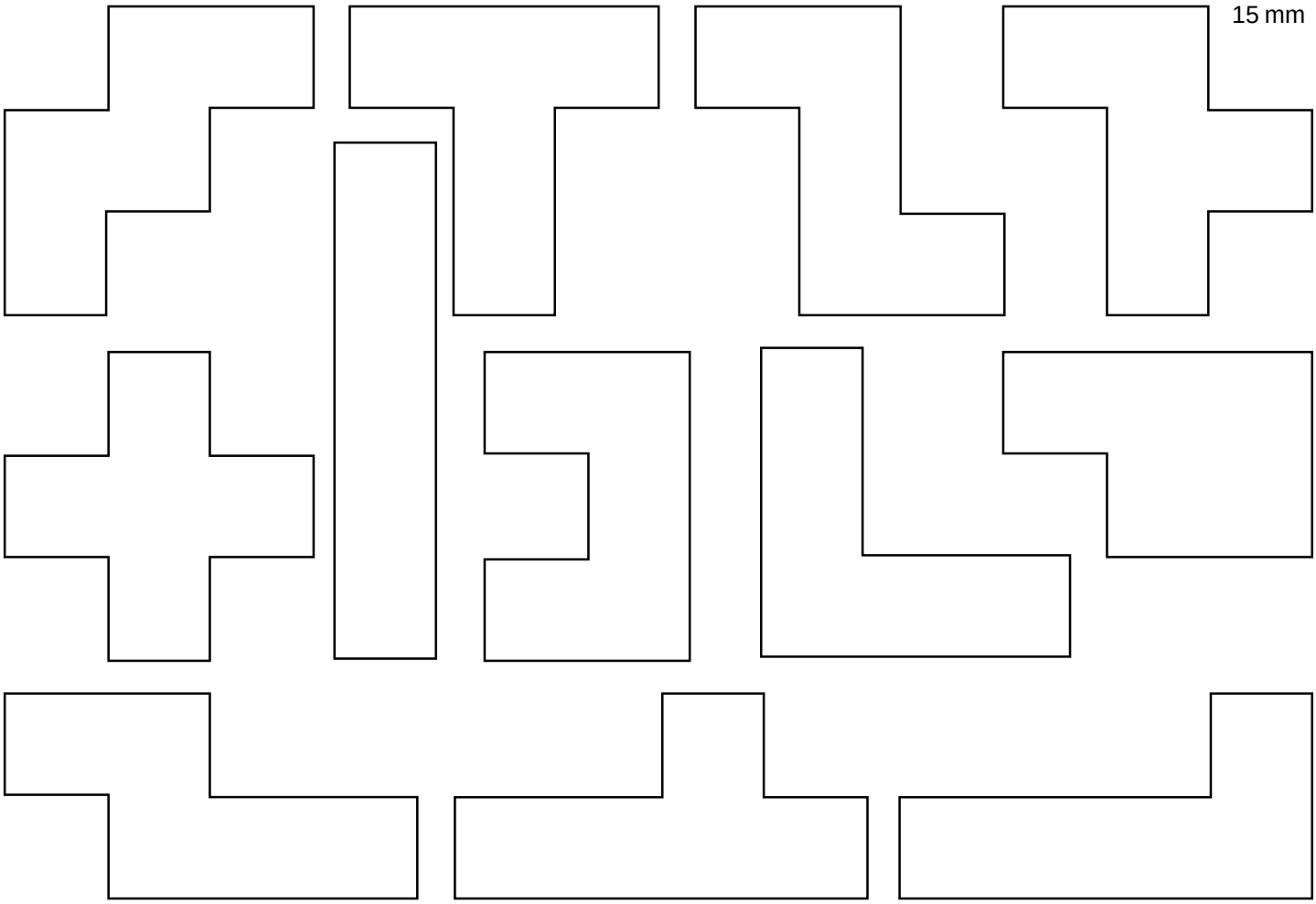


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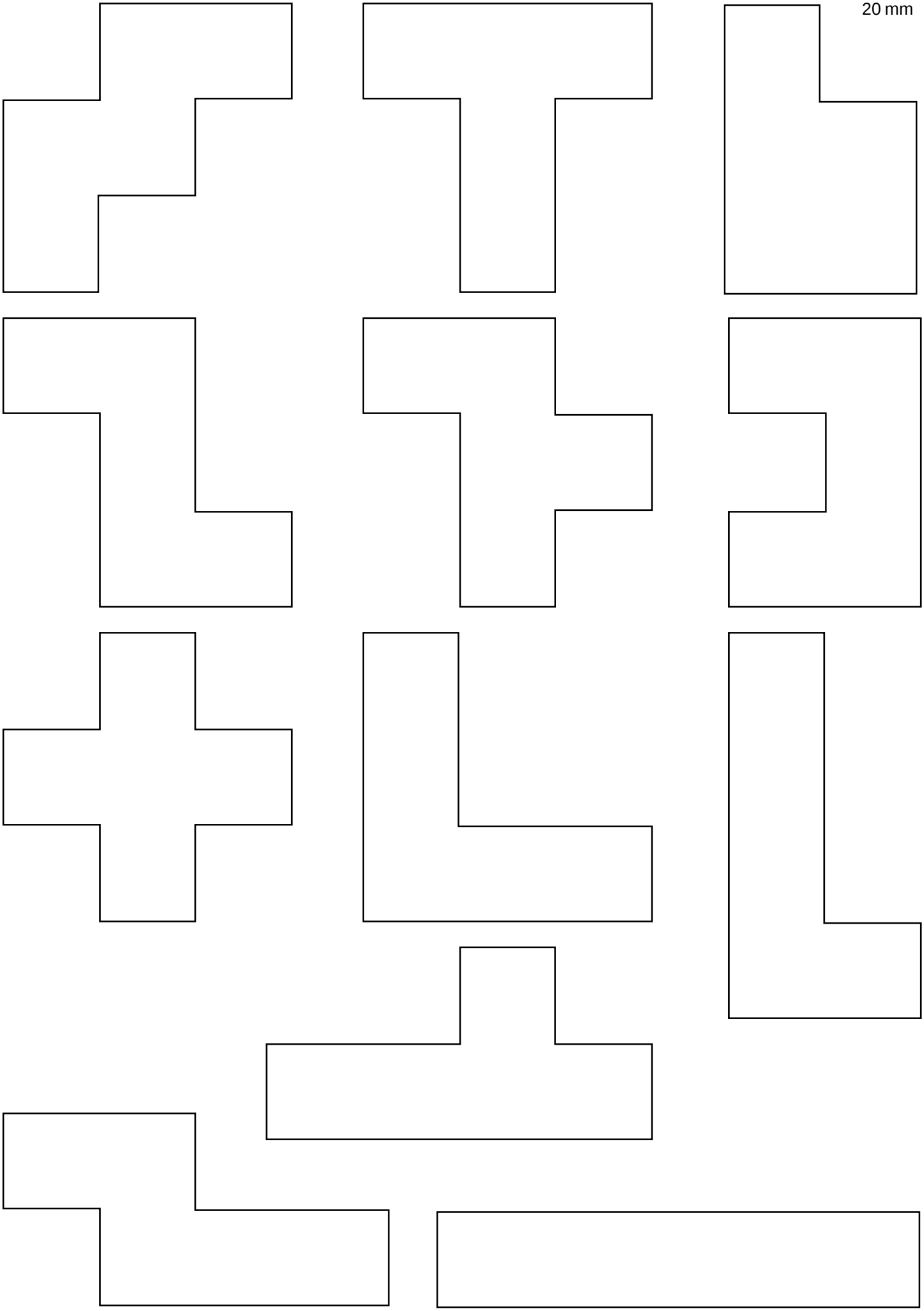
20 mm



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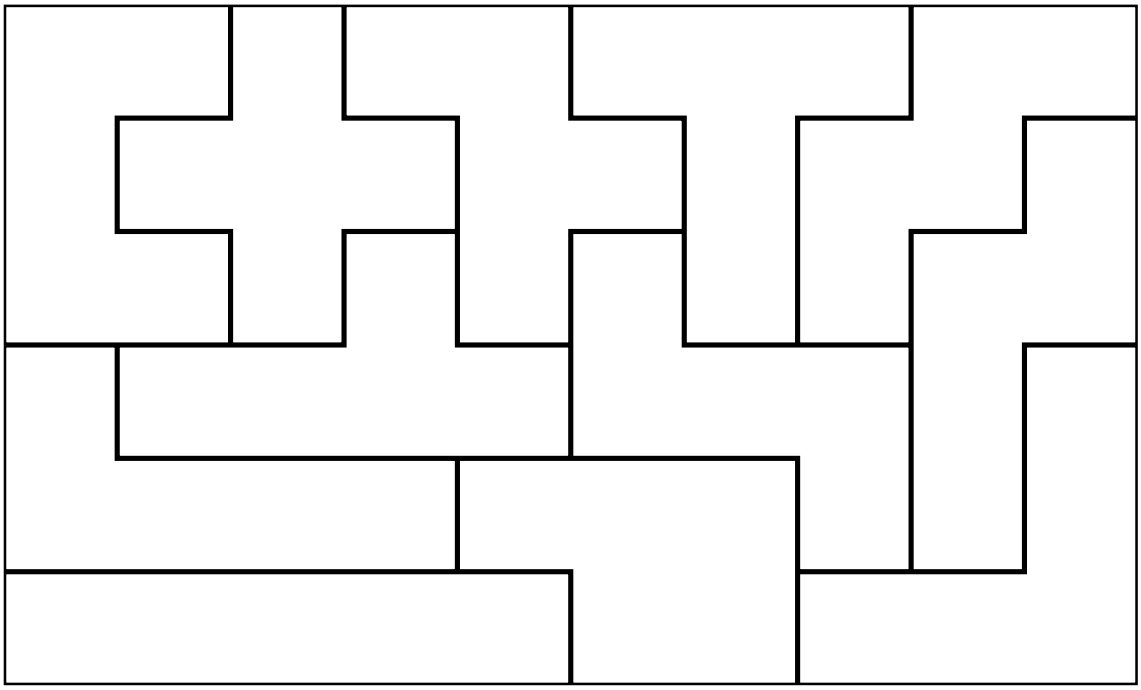

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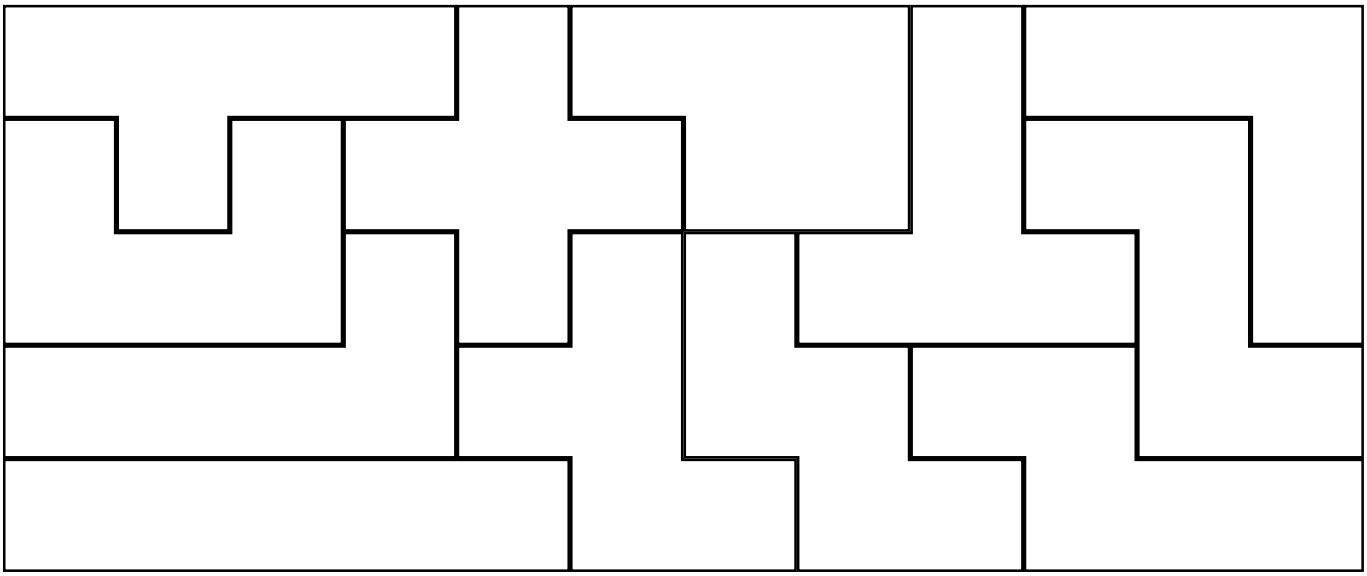

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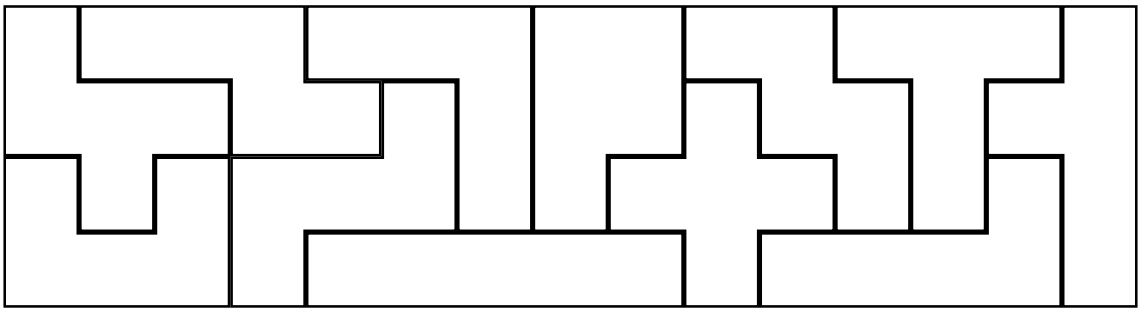

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